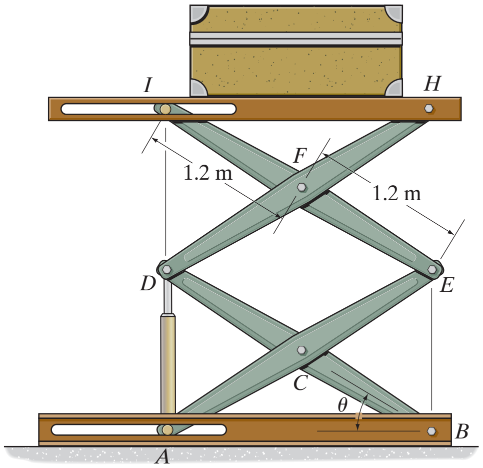
•11–1. The 200-kg crate is on the lift table at the position $\theta = 30°$. Determine the force in the hydraulic cylinder AD for equilibrium. Neglect the mass of the lift table's components.



Free - Body Diagram: When θ undergoes a positive virtual displacement of $\delta\theta$, the dash line configuration shown in Fig. *a* is formed. We observe that only the force in hydraulic cylinder $\mathbf{F}_{AD}$ acting at point D and the weight of the crate $\mathbf{W}_J$ do work when the virtual displacements take place.

Virtual Displacement: The position of $\mathbf{F}_{AD}$ acting at point D and the point of application of $\mathbf{W}_J$ are specified by the position coordinates y_D and y_J, measured from the fixed point B.

$$y_D = 2.4\sin\theta \qquad \delta y_D = 2.4\cos\theta\delta\theta \qquad (1)$$

$$y_J = 2(2.4\sin\theta) + b \qquad \delta y_J = 4.8\cos\theta\delta\theta \qquad (2)$$

Virtual Work Equation: Since $\mathbf{F}_{AD}$ acts towards the positive sense of its corresponding virtual displacement, its work is positive. The work of $\mathbf{W}_J$ is negative since it acts towards the negative sense of its corresponding virtual displacement.

$$\delta U = 0; \qquad F_{AD}\delta y_D + [-200(9.81)\delta y_J] = 0 \qquad (3)$$

Substituting Eqs. (1) and (2) into Eq. (3),

$$F_{AD}(2.4\cos\theta\delta\theta) - 200(9.81)(4.8\cos\theta\delta\theta) = 0$$

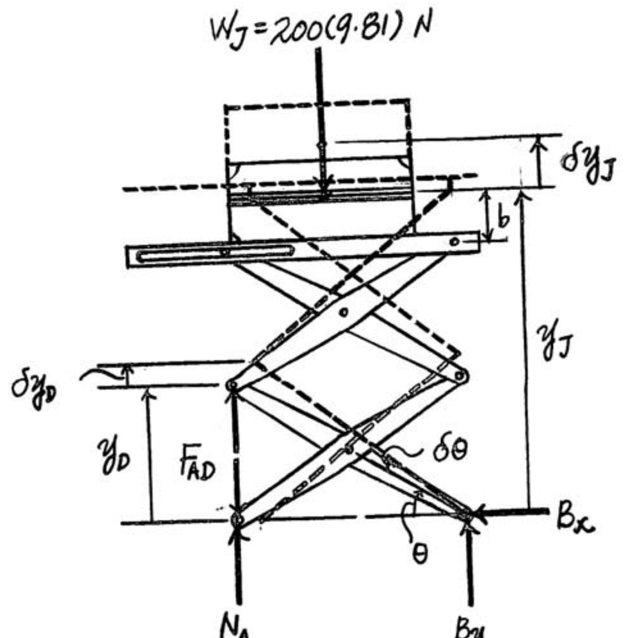
$$\cos\theta\delta\theta(2.4F_{AD} - 9417.6) = 0$$

Since $\cos\theta\delta\theta \neq 0$, then

$$2.4F_{AD} - 9417.6 = 0$$

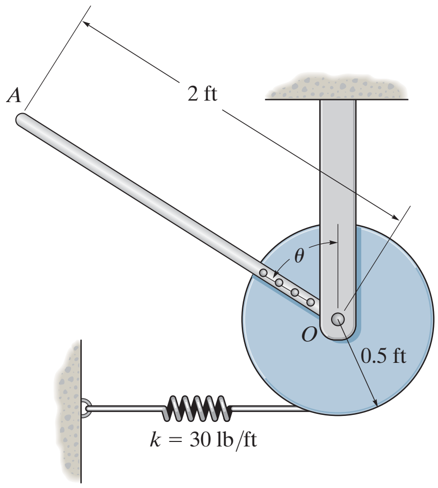
$$F_{AD} = 3924 \text{ N} = 3.92 \text{ kN} \qquad \textbf{Ans.}$$

Note: $\mathbf{F}_{AD}$ remains constant regardless of angle θ.



(a)

11–2. The uniform rod OA has a weight of 10 lb. When the rod is in a vertical position, $\theta = 0°$, the spring is unstretched. Determine the angle θ for equilibrium if the end of the spring wraps around the periphery of the disk as the disk turns.



Free Body Diagram : The system has only one degree of freedom defined by the independent coordinate θ. When θ undergoes a positive displacement $\delta\theta$, only the spring force and the weight of rod (10 lb force) do work.

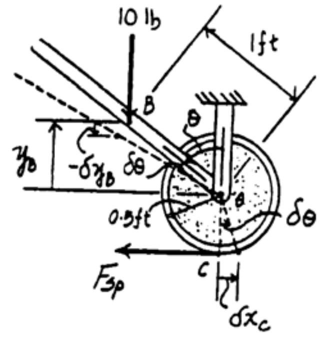
Virtual Displacements : The 10 lb force is located from the fixed point O using the position coordinate y_B, and the virtual displacement of point C is δx_C.

$$y_B = 1\cos\theta \qquad \delta y_B = -\sin\theta\,\delta\theta \qquad [1]$$

$$\delta x_C = 0.5\delta\theta \qquad [2]$$

Virtual - Work Equation : When points B and C undergo positive virtual displacements δy_B and δx_C, the 10 lb force and the spring force F_{sp}, do negative work.

$$\delta U = 0; \quad -10\delta y_B - F_{sp}\,\delta x_C = 0 \qquad [3]$$



Substituting Eqs. [1] and [2] into [3] yields

$$\left(-10\sin\theta - 0.5F_{sp}\right)\delta\theta = 0 \qquad [4]$$

However, from the spring formula, $F_{sp} = kx = 30(0.5\theta) = 15\theta$. Substituting this value into Eq. [4] yields

$$(-10\sin\theta - 7.5\theta)\,\delta\theta = 0$$

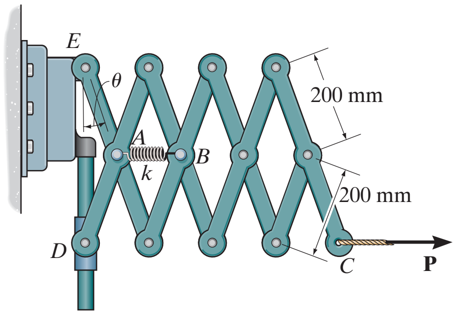
Since $\delta\theta \neq 0$, then

$$-10\sin\theta - 7.5\theta = 0$$

Solving by trial and error

$$\theta = 0° \quad \text{and} \quad \theta = 73.1° \qquad \textbf{Ans}$$

11–3. The "Nuremberg scissors" is subjected to a horizontal force of $P = 600$ N. Determine the angle θ for equilibrium. The spring has a stiffness of $k = 15$ kN/m and is unstretched when $\theta = 15°$.



Free - Body Diagram: When θ undergoes a positive virtual angular displacement of $\delta\theta$, the dash line configuration shown in Fig. *a* is formed. We observe that only the spring force $\mathbf{F}_{sp}$ acting at points A and B and the force $\mathbf{P}$ do work when the virtual displacements take place. The magnitude of $\mathbf{F}_{sp}$ can be computed using the spring force formula,

$$F_{sp} = kx = 15(10^3)[2(0.2\sin\theta) - 2(0.2\sin 15°)] = 6000(\sin\theta - 0.2588)\text{ N}.$$

Virtual Displacement: The position of points A and B at which $\mathbf{F}_{sp}$ acts and point C at which force $\mathbf{P}$ acts are specified by the position coordinates y_A, y_B, and y_C, measured from the fixed point E, respectively.

$$y_A = 0.2\sin\theta \qquad \delta y_A = 0.2\cos\theta\delta\theta \qquad (1)$$
$$y_B = 3(0.2\sin\theta) \qquad \delta y_B = 0.6\cos\theta\delta\theta \qquad (2)$$
$$y_C = 8(0.2\sin\theta) \qquad \delta y_B = 1.6\cos\theta\delta\theta \qquad (3)$$

Virtual Work Equation: Since $\mathbf{F}_{sp}$ at point A and force $\mathbf{P}$ acts towards the positive sense of its corresponding virtual displacement, their work is positive. The work of $\mathbf{F}_{sp}$ at point B is negative since it acts towards the negative sense of its corresponding virtual displacement. Thus,

$$\delta U = 0; \qquad F_{sp}\delta y_A + \left(-F_{sp}\delta y_B\right) + P\delta y_C = 0 \qquad (4)$$

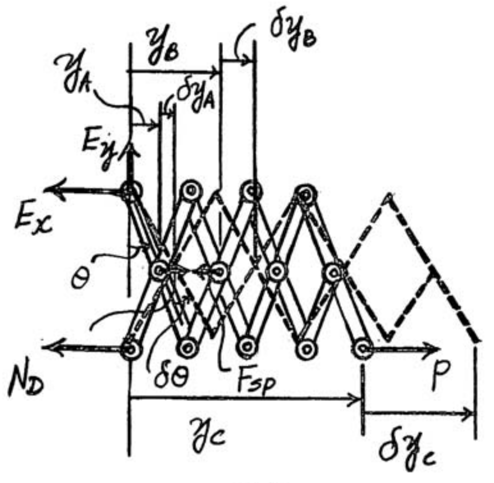
Substituting $F_{sp} = 6000(\sin\theta - 0.2588)$, $P = 600$ N, Eqs. (1), (2), and (3) into Eq. (4),

$$6000(\sin\theta - 0.2588)(0.2\cos\theta\delta\theta - 0.6\cos\theta\delta\theta) + 600(1.6\cos\theta\delta\theta) = 0$$
$$\cos\theta\delta\theta[-2400(\sin\theta - 0.2588) + 960] = 0$$

Since $\cos\theta\delta\theta \neq 0$, then

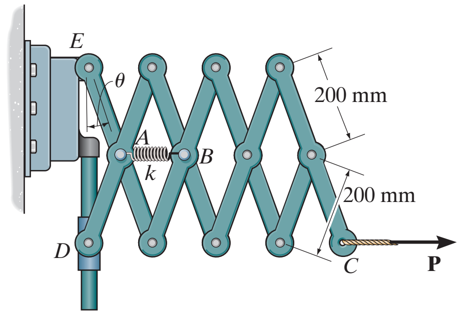
$$-2400(\sin\theta - 0.2588) + 960 = 0$$
$$\theta = 41.2°$$

Ans.



(a)

***11–4.** The "Nuremberg scissors" is subjected to a horizontal force of $P = 600$ N. Determine the stiffness k of the spring for equilibrium when $\theta = 60°$. The spring is unstretched when $\theta = 15°$.



Free - Body Diagram: When θ undergoes a positive virtual angular displacement of $\delta\theta$, the dash line configuration shown in Fig. *a* is formed. We observe that only the spring force $\mathbf{F}_{sp}$ acting at points A and B and the force **P** do work when the virtual displacements take place. The magnitude of $\mathbf{F}_{sp}$ can be computed using the spring force formula.

$$F_{sp} = kx = k[2(0.2\sin\theta) - 2(0.2\sin 15°)] = (0.4)k(\sin\theta - 0.2588)\text{ N}$$

Virtual Displacement: The position of points A and B at which $\mathbf{F}_{sp}$ acts and point C at which force **P** acts are specified by the position coordinates y_A, y_B, and y_C, measured from the fixed point E, respectively.

$$y_A = 0.2\sin\theta \qquad \delta y_A = 0.2\cos\theta\delta\theta \qquad (1)$$

$$y_B = 3(0.2\sin\theta) \qquad \delta y_B = 0.6\cos\theta\delta\theta \qquad (2)$$

$$y_C = 8(0.2\sin\theta) \qquad \delta y_B = 1.6\cos\theta\delta\theta \qquad (3)$$

Virtual Work Equation: Since $\mathbf{F}_{sp}$ at point A and force **P** acts towards the positive sense of its corresponding virtual displacement, their work is positive. The work of $\mathbf{F}_{sp}$ at point B is negative since it acts towards the negative sense of its corresponding virtual displacement. Thus,

$$\delta U = 0; \qquad F_{sp}\delta y_A + \left(-F_{sp}\delta y_B\right) + P\delta y_C = 0 \qquad (4)$$

Substituting $F_{sp} = k(\sin\theta - 0.2588)$, $P = 600$ N, Eqs. (1), (2), and (3) into Eq. (4),

$$(0.4)k(\sin\theta - 0.2588)(0.2\cos\theta\delta\theta - 0.6\cos\theta\delta\theta) + 600(1.6\cos\theta\delta\theta) = 0$$

$$\cos\theta\delta\theta[-0.16k(\sin\theta - 0.2588) + 960] = 0$$

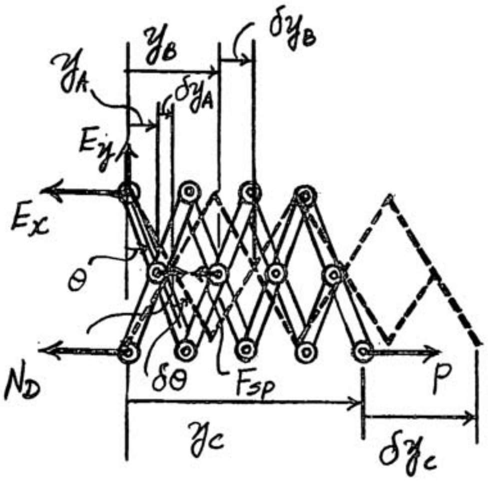
Since $\cos\theta\delta\theta \neq 0$, then

$$-0.16k(\sin\theta - 0.2588) + 960 = 0$$

$$k = \frac{6000}{\sin\theta - 0.2588}$$

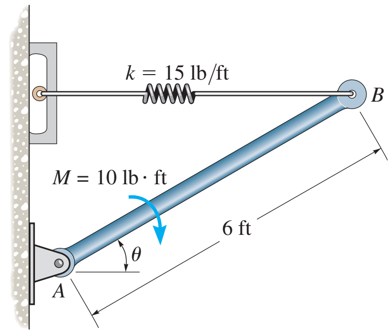
When $\theta = 60°$,

$$k = \frac{6000}{\sin 60° - 0.2588} = 9881\text{ N/m} = 9.88\text{ kN/m} \qquad \textbf{Ans.}$$



(a)

•11–5. Determine the force developed in the spring required to keep the 10 lb uniform rod AB in equilibrium when $\theta = 35°$.



Free - Body Diagram : The system has only one degree of freedom defined by the independent coordinate θ. When θ undergoes a positive displacement $\delta\theta$, only the spring force F_{sp}, the weight of the rod (10 lb) and the 10 lb · ft couple moment do work.

Virtual Displacements : The spring force F_{sp} and the weight of the rod (10 lb) are located from the fixed point A using position coordinates x_B and x_C, respectively.

$$x_B = 6\cos\theta \qquad \delta x_B = -6\sin\theta\,\delta\theta \qquad [1]$$
$$y_C = 3\sin\theta \qquad \delta y_C = 3\cos\theta\,\delta\theta \qquad [2]$$

Virtual - Work Equation : When points B and C undergo positive virtual displacements δx_B and δy_C, the spring force F_{sp} and the weight of the rod (10 lb) do negative work. The 10 lb · ft couple moment does negative work when rod AB undergoes a positive virtual rotation $\delta\theta$.

$$\delta U = 0; \qquad -F_{sp}\,\delta x_B - 10\delta y_C - 10\delta\theta = 0 \qquad [3]$$

Substituting Eqs. [1] and [2] into [3] yields

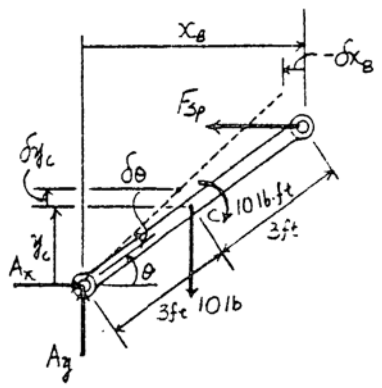
$$\left(6F_{sp}\sin\theta - 30\cos\theta - 10\right)\delta\theta = 0 \qquad [4]$$

Since $\delta\theta \neq 0$, then

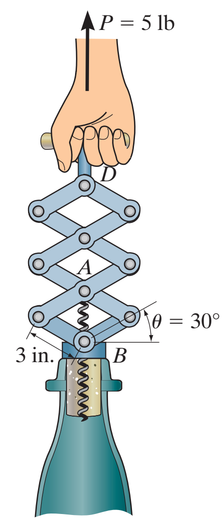
$$6F_{sp}\sin\theta - 30\cos\theta - 10 = 0$$
$$F_{sp} = \frac{30\cos\theta + 10}{6\sin\theta}$$

At the equilibrium position, $\theta = 35°$. Then

$$F_{sp} = \frac{30\cos 35° + 10}{6\sin 35°} = 10.0 \text{ lb} \qquad \textbf{Ans}$$



11–6. If a force of $P = 5$ lb is applied to the handle of the mechanism, determine the force the screw exerts on the cork of the bottle. The screw is attached to the pin at A and passes through the collar that is attached to the bottle neck at B.



Free - Body Diagram: When θ undergoes a positive virtual angular displacement of $\delta\theta$, the dash line configuration shown in Fig. *a* is formed. We observe that only the force in the screw $\mathbf{F}_s$ and force $\mathbf{P}$ do work when the virtual displacements take place.

Virtual Displacement: The position of the points of application for $\mathbf{F}_s$ and $\mathbf{P}$ are specified by the position coordinates y_A and y_D, measured from the fixed point B, respectively.

$$y_A = 2(3\sin\theta) \qquad \delta y_A = 6\cos\theta\delta\theta \qquad (1)$$

$$y_D = 6(3\sin\theta) \qquad \delta y_D = 18\cos\theta\delta\theta \qquad (2)$$

Virtual Work Equation: Since $\mathbf{P}$ acts towards the positive sense of its corresponding virtual displacement, it does positive work. However, the work of $\mathbf{F}_s$ is negative since it acts towards the negative sense of its corresponding virtual displacement. Thus,

$$\delta U = 0; \qquad P\delta y_D + (-F_s\delta y_A) = 0 \qquad (3)$$

Substituting $P = 5$ lb, Eqs. (1) and (2) into Eq. (3),

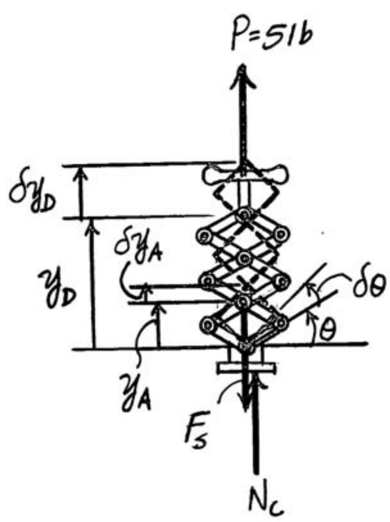
$$5(18\cos\theta\delta\theta) - F_s(6\cos\theta\delta\theta) = 0$$

$$\cos\theta\delta\theta(90 - 6F_s) = 0$$

Since $\cos\theta\delta\theta \neq 0$, then

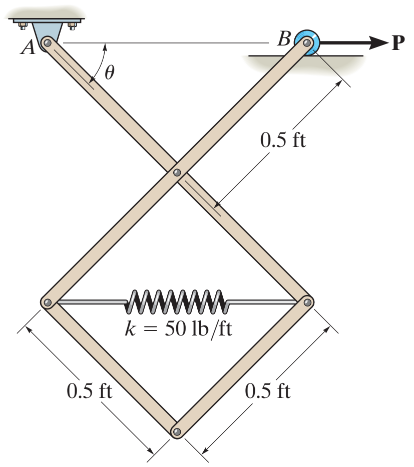
$$90 - 6F_s = 0$$

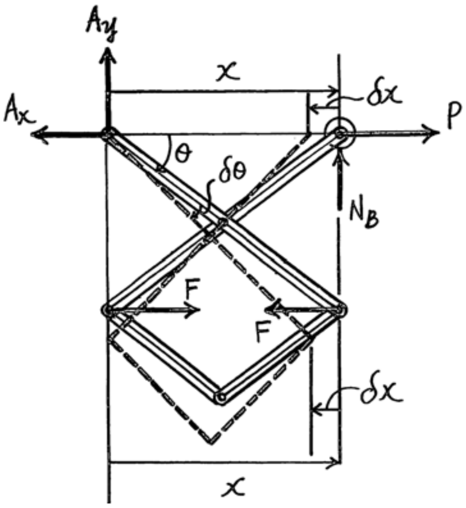
$$F_s = 15 \text{ lb} \qquad \textbf{Ans.}$$



(a)

11–7. The pin-connected mechanism is constrained at A by a pin and at B by a roller. If $P = 10$ lb, determine the angle θ for equilibrium. The spring is unstretched when $\theta = 45°$. Neglect the weight of the members.





$x = l\cos\theta$

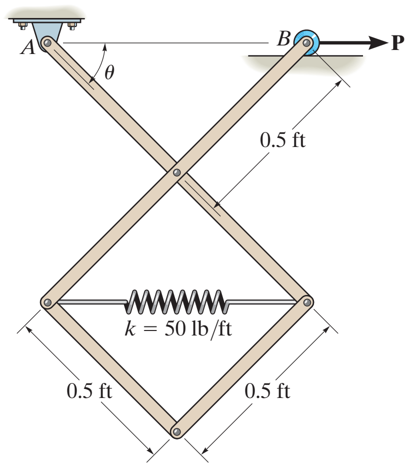
$F_s = ks; \quad F = 50(l\cos\theta - l\cos 45°)$

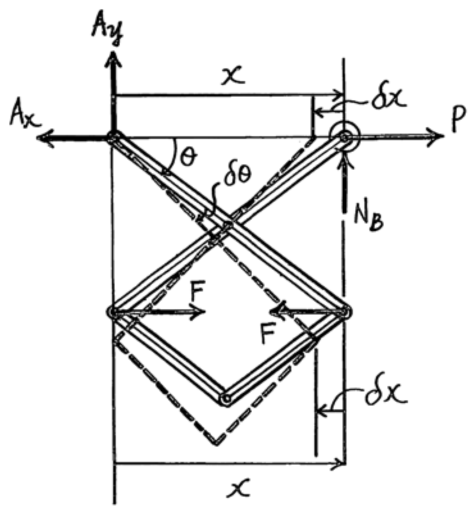
$\delta U = 0; \quad -F\,\delta x + P\,\delta x = 0$

$-50(l\cos\theta - l\cos 45°) + 10 = 0$

$\theta = 24.9°$ **Ans**

*11–8. The pin-connected mechanism is constrained by a pin at A and a roller at B. Determine the force P that must be applied to the roller to hold the mechanism in equilibrium when $\theta = 30°$. The spring is unstretched when $\theta = 45°$. Neglect the weight of the members.





$x = l\cos\theta$

$\delta U = 0; \quad P\,\delta x - F_s\,\delta x = 0$

$P = F_s$

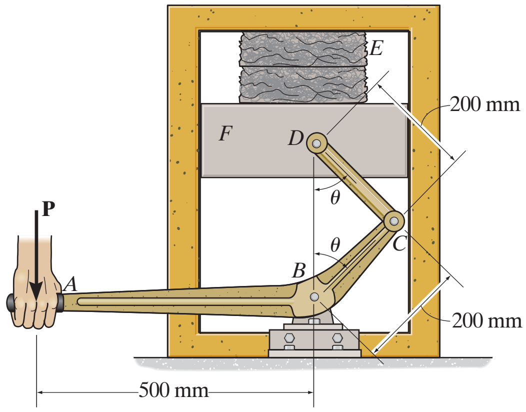
When $\theta = 45°$, $x = l\cos 45° = 0.7071$ ft

When $\theta = 30°$, $x = l\cos 30° = 0.86602$ ft

$F_s = kx; \quad F_s = 50(0.86602 - 0.7071) = 7.95$ lb

$P = 7.95$ lb **Ans**

•11–9. If a force $P = 100$ N is applied to the lever arm of the toggle press, determine the clamping force developed in the block when $\theta = 45°$. Neglect the weight of the block.



Free - Body Diagram: When θ undergoes a positive virtual angular displacement of $\delta\theta$, the dash line configuration shown in Fig. *a* is formed. We observe that only force **P** and the clamping force $\mathbf{F}_E$ do work when the virtual displacement takes place.

Virtual Displacement: The position of point D at which $\mathbf{F}_E$ acts is specified by the position coordinate y_D.

$$y_D = 2(0.2\cos\theta) \qquad \delta y_D = -0.4\sin\theta\delta\theta \qquad (1)$$

Since $\delta\theta$ is very small, the virtual displacement of point A at which force **P** acts is

$$\delta y_A = 0.5\delta\theta \qquad (2)$$

Virtual Work Equation: Since $\mathbf{F}_E$ and **P** act towards the negative sense of their corresponding virtual displacements, their work is negative. Thus,

$$\delta U = 0; \qquad -P\delta y_A + (-F_E\delta y_D) = 0 \qquad (3)$$

Substituting $P = 100$ N, Eqs. (1) and (2) into Eq. (3),

$$-100(0.5\delta\theta) - F_E(-0.4\sin\theta\delta\theta) = 0$$

$$\delta\theta(-50 + 0.4F_E\sin\theta) = 0$$

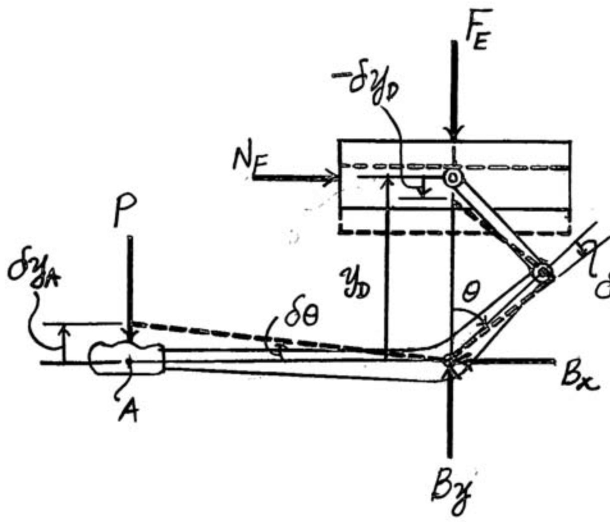
Since $\delta\theta \neq 0$, then

$$-50 + 0.4F_E\sin\theta = 0$$

$$F_E = \frac{125}{\sin\theta}$$

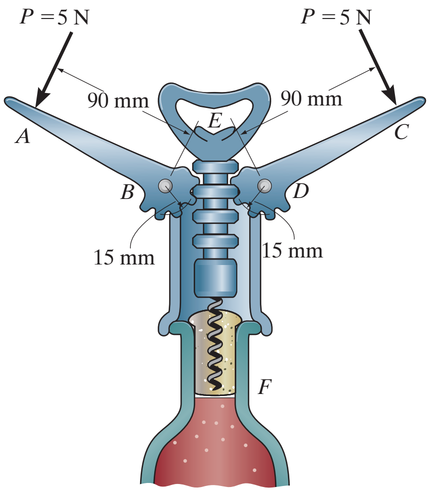
At $\theta = 45°$,

$$F_E = \frac{125}{\sin 45°} = 176.78\text{ N} = 177\text{ N} \qquad \textbf{Ans.}$$



(a)

11–10. When the forces are applied to the handles of the bottle opener, determine the pulling force developed on the cork.



Free - Body Diagram: When the handle undergoes a virtual angular displacement of $\delta\theta$, only forces **P** and **F** do work, Fig. *a*.

Virtual Displacement: Since $\delta\theta$ is very small, the virtual displacements of forces **P** and **F** can be approximated as

$$\delta_P = 0.09\delta\theta \qquad (1)$$

$$\delta_F = 0.015\delta\theta \qquad (2)$$

Virtual Work Equation: Since **P** acts towards the positive sense of its corresponding virtual displacement, its work is positive. However, force **F** does negative work since it acts towards the negative sense.

$$\delta U = 0; \qquad 2(P\delta_P) + (-F\delta F) = 0 \qquad (3)$$

Substituting $P = 5$ N and Eqs. (1) and (2) into Eq. (3),

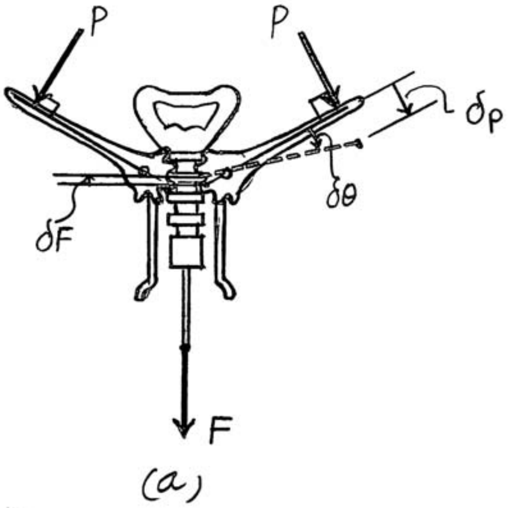
$$2[5(0.09\delta\theta)] - F(0.015\delta\theta) = 0$$

$$\delta\theta(0.9 - 0.015F) = 0$$

Since $\delta\theta \neq 0$, then

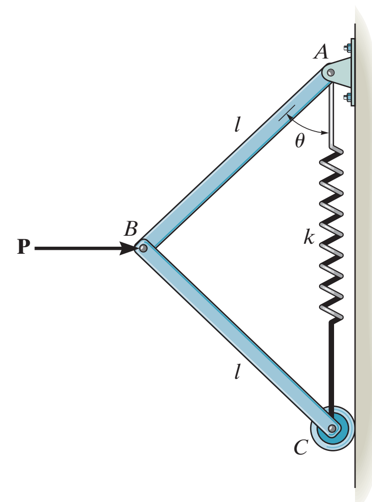
$$0.9 - 0.015F = 0$$

$$F = 60 \text{ N} \qquad \textbf{Ans.}$$



(a)

11–11. If the spring has a stiffness k and an unstretched length l_0, determine the force P when the mechanism is in the position shown. Neglect the weight of the members.



$y = 2l\cos\theta, \quad \delta y = -2l\sin\theta\,\delta\theta$

$x = l\sin\theta, \quad \delta x = l\cos\theta\,\delta\theta$

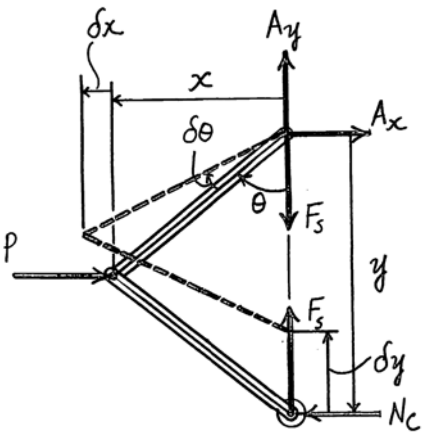
$\delta U = 0; \quad -P\,\delta x - F_s\,\delta y = 0$

$-P(l\cos\theta\,\delta\theta) + F_s(2l\sin\theta\,\delta\theta) = 0$

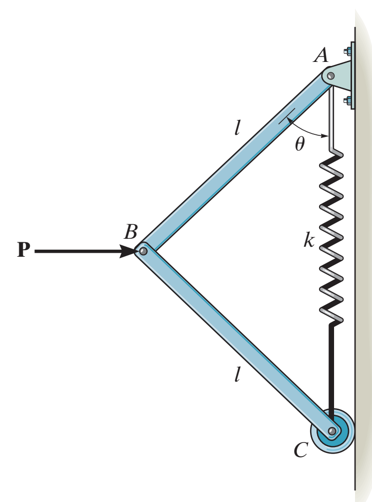
$-P\cos\theta + 2F_s\sin\theta = 0$

$F_s = k(2l\cos\theta - l_0)$

$P = 2k\tan\theta\,(2l\cos\theta - l_0)$ **Ans**



***11–12.** Solve Prob. 11–11 if the force **P** is applied vertically downward at B.



$y_1 = l\cos\theta, \quad \delta y_1 = -l\sin\theta\,\delta\theta$

$y_2 = 2l\cos\theta, \quad \delta y_2 = -2l\sin\theta\,\delta\theta$

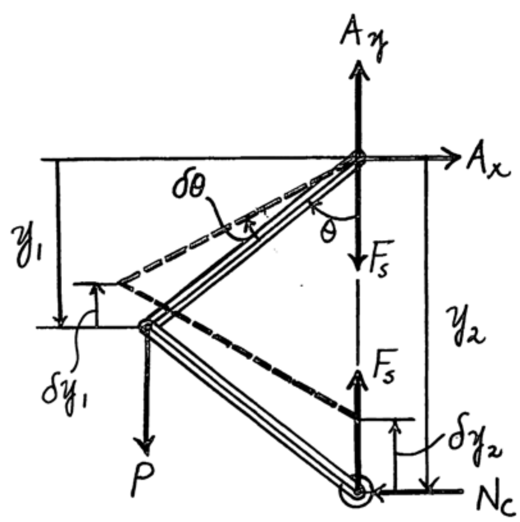
$\delta U = 0; \quad P\,\delta y_1 - F_s\,\delta y_2 = 0$

$P(-l\sin\theta)\,\delta\theta - F_s(-2l\sin\theta)\,\delta\theta = 0$

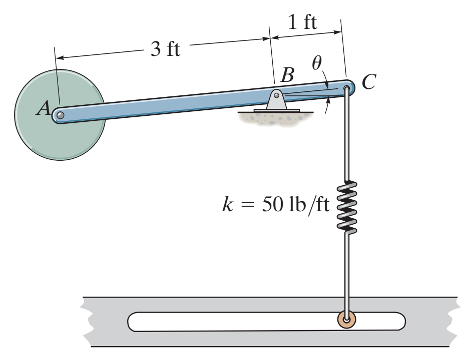
$P = 2F_s$

$F_s = ks; \quad F_s = k(2l\cos\theta - l_0)$

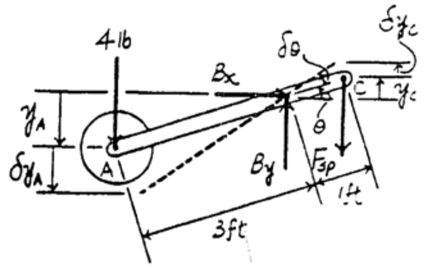
$P = 2k(2l\cos\theta - l_0)$ **Ans**



•11–13. Determine the angles θ for equilibrium of the 4-lb disk using the principle of virtual work. Neglect the weight of the rod. The spring is unstretched when $\theta = 0°$ and always remains in the vertical position due to the roller guide.



Free Body Diagram : The system has only one degree of freedom defined by the independent coordinate θ. When θ undergoes a positive displacement $\delta\theta$, only the spring force F_{sp} and the weight of the disk (4 lb) do work.



Virtual Displacements : The spring force F_{sp} and the weight of the disk (4 lb) are located from the fixed point B using position coordinates y_C and y_A, respectively.

$$y_C = 1\sin\theta \qquad \delta y_C = \cos\theta\delta\theta \qquad [1]$$
$$y_A = 3\sin\theta \qquad \delta y_A = 3\cos\theta\delta\theta \qquad [2]$$

Virtual - Work Equation : When points C and A undergo positive virtual displacements δy_C and δy_A, the spring force F_{sp} does negative work while the weight of the disk (4 lb) do positive work.

$$\delta U = 0; \qquad 4\delta y_A - F_{sp}\delta y_C = 0 \qquad [3]$$

Substituting Eqs.[1] and [2] into [3] yields

$$\left(12 - F_{sp}\right)\cos\theta\delta\theta = 0 \qquad [4]$$

However, from the spring formula, $F_{sp} = kx = 50(1\sin\theta) = 50\sin\theta$. Substituting this value into Eq.[4] yields

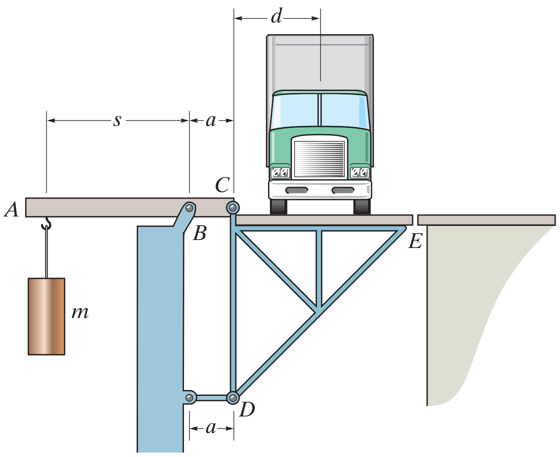
$$(12 - 50\sin\theta)\cos\theta\delta\theta = 0$$

Since $\delta\theta \neq 0$, then

$$12 - 50\sin\theta = 0 \qquad \theta = 13.9° \qquad \textbf{Ans}$$

$$\cos\theta = 0 \qquad \theta = 90° \qquad \textbf{Ans}$$

11–14. The truck is weighed on the highway inspection scale. If a known mass m is placed a distance s from the fulcrum B of the scale, determine the mass of the truck m_t if its center of gravity is located at a distance d from point C. When the scale is empty, the weight of the lever ABC balances the scale CDE.



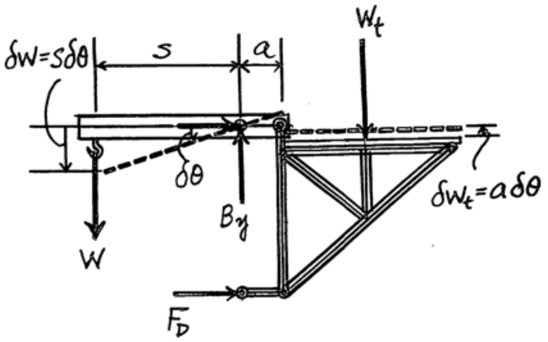
$\delta U = 0;$ $\quad (W)(s\,\delta\theta) - W_t a\,\delta\theta = 0$

$(Ws - W_t a)\delta\theta = 0$

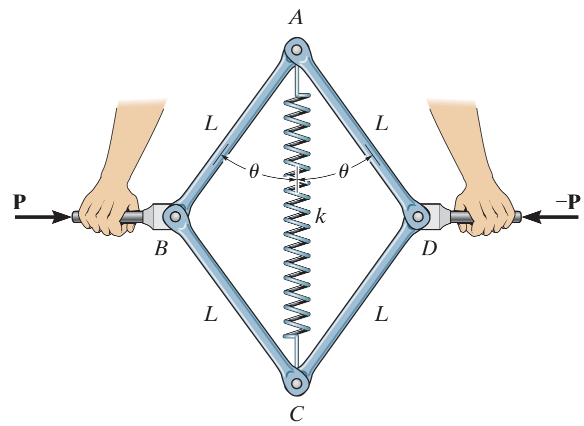
$W_t = W\left(\frac{s}{a}\right)$

or,

$m_t = m\left(\frac{s}{a}\right)$ **Ans**



11–15. The assembly is used for exercise. It consists of four pin-connected bars, each of length L, and a spring of stiffness k and unstretched length $a\,(< 2L)$. If horizontal forces are applied to the handles so that θ is slowly decreased, determine the angle θ at which the magnitude of **P** becomes a maximum.



Free Body Diagram : The system has only one degree of freedom defined by the independent coordinate θ. When θ undergoes a positive displacement $\delta\theta$, the spring force F_{sp} and force **P** do work.

Virtual Displacements : The spring force F_{sp} and force **P** are located from the fixed point D and A using position coordinates y and x, respectively.

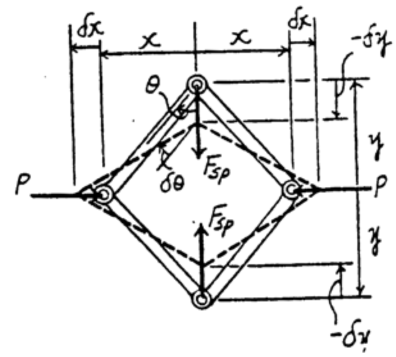
$$y = L\cos\theta \qquad \delta y = -L\sin\theta\,\delta\theta \qquad [1]$$
$$x = L\sin\theta \qquad \delta x = L\cos\theta\,\delta\theta \qquad [2]$$

Virtual - Work Equation : When points A, C, B and D undergo positive virtual displacement δy and δx, the spring force F_{sp} and force **P** do negative work.

$$\delta U = 0; \qquad -2F_{sp}\,\delta y - 2P\delta x = 0 \qquad [3]$$

Substituting Eqs. [1] and [2] into [3] yields

$$\left(2F_{sp}\sin\theta - 2P\cos\theta\right)L\delta\theta = 0 \qquad [4]$$



From the geometry, the spring stretches $\zeta = 2L\cos\theta - a$. Then, the spring force $F_{sp} = k\zeta = k(2L\cos\theta - a) = 2kL\cos\theta - ka$. Substituting this value into Eq. [4] yields

$$(4kL\sin\theta\cos\theta - 2ka\sin\theta - 2P\cos\theta)\,L\delta\theta = 0$$

Since $L\delta\theta \neq 0$, then

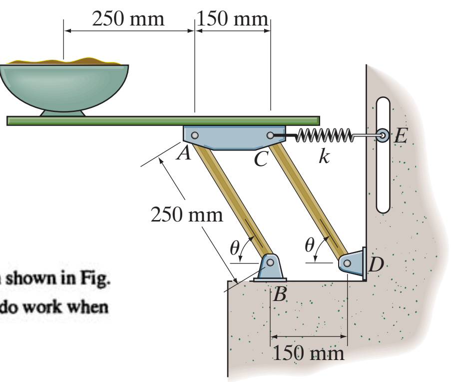
$$4kL\sin\theta\cos\theta - 2ka\sin\theta - 2P\cos\theta = 0$$
$$P = k(2L\sin\theta - a\tan\theta)$$

In order to obtain maximum P, $\dfrac{dP}{d\theta} = 0$.

$$\frac{dP}{d\theta} = k\left(2L\cos\theta - a\sec^2\theta\right) = 0$$

$$\theta = \cos^{-1}\left(\frac{a}{2L}\right)^{\frac{1}{3}} \qquad \textbf{Ans}$$

***11–16.** A 5-kg uniform serving table is supported on each side by pairs of two identical links, AB and CD, and springs CE. If the bowl has a mass of 1 kg, determine the angle θ where the table is in equilibrium. The springs each have a stiffness of $k = 200$ N/m and are unstretched when $\theta = 90°$. Neglect the mass of the links.



Free - Body Diagram: When θ undergoes a positive virtual angular displacement of $\delta\theta$, the dash line configuration shown in Fig. a is formed. We observe that only the spring force $\mathbf{F}_{sp}$, the weight $\mathbf{W}_t$ of the table, and the weight $\mathbf{W}_b$ of the bowl do work when the virtual displacement takes place. The magnitude of $\mathbf{F}_{sp}$ can be computed using the spring force formula,

$F_{sp} = kx = 200(0.25\cos\theta) = 50\cos\theta$ N.

Virtual Displacement: The position of points of application of $\mathbf{W}_b$, $\mathbf{W}_t$, and $\mathbf{F}_{sp}$ are specified by the position coordinates y_{G_b}, y_{G_t}, and x_C, respectively. Here, y_{G_b} and y_{G_t} are measured from the fixed point B while x_C is measured from the fixed point D.

$$y_{G_b} = 0.25\sin\theta + b \qquad \delta y_{G_b} = 0.25\cos\theta\delta\theta \qquad (1)$$

$$y_{G_t} = 0.25\sin\theta + a \qquad \delta y_{G_t} = 0.25\cos\theta\delta\theta \qquad (2)$$

$$x_C = 0.25\cos\theta \qquad \delta x_C = -0.25\sin\theta\delta\theta \qquad (3)$$

Virtual Work Equation: Since $\mathbf{W}_b$, $\mathbf{W}_t$, and $\mathbf{F}_{sp}$ act towards the negative sense of their corresponding virtual displacement, their work is negative. Thus,

$$\delta U = 0; \qquad -W_b\delta y_{G_b} + \left(-W_t\delta y_{G_t}\right) + \left(-F_{sp}\delta x_C\right) = 0 \quad (4)$$

Substituting $W_b = \left(\frac{1}{2}\right)(9.81) = 4.905$ N, $W_t = \left(\frac{5}{2}\right)(9.81) = 24.525$ N, $F_{sp} = 50\cos\theta$ N, Eqs. (1), (2), and (3) into Eq. (4), we have

$$-4.905(0.25\cos\theta\delta\theta) - 24.525(0.25\cos\theta\delta\theta) - 50\cos\theta(-0.25\sin\theta\delta\theta) = 0$$

$$\delta\theta(-7.3575\cos\theta + 12.5\sin\theta\cos\theta) = 0$$

Since $\delta\theta \neq 0$, then

$$-7.3575\cos\theta + 12.5\sin\theta\cos\theta = 0$$

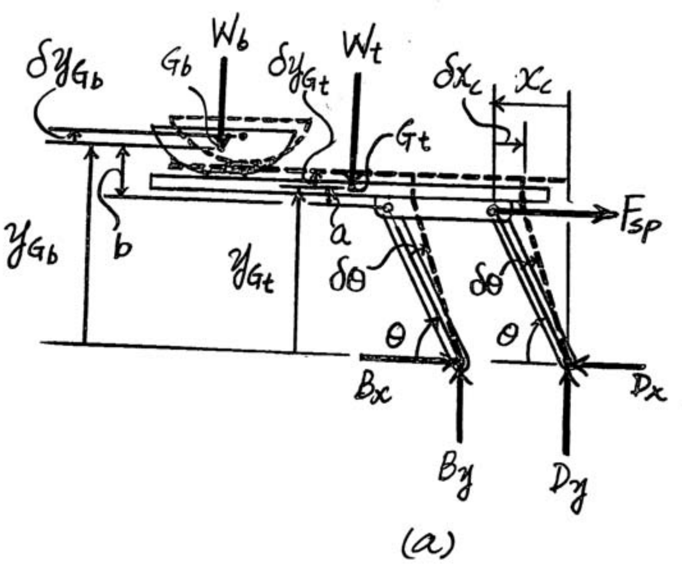
$$\cos\theta(-7.3575 + 12.5\sin\theta) = 0$$

Solving the above equation,

$\cos\theta = 0 \quad \theta = 90°$ **Ans.**

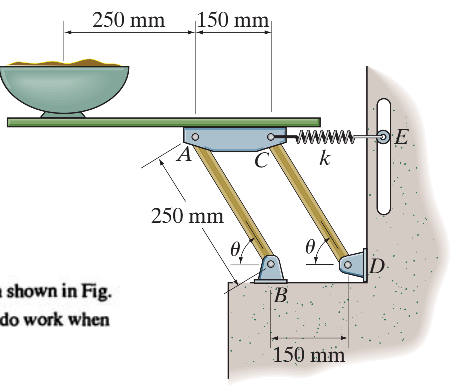
$-7.3575 + 12.5\sin\theta = 0$

$\theta = 36.1°$ **Ans.**



(a)

•11–17. A 5-kg uniform serving table is supported on each side by two pairs of identical links, AB and CD, and springs CE. If the bowl has a mass of 1 kg and is in equilibrium when $\theta = 45°$, determine the stiffness k of each spring. The springs are unstretched when $\theta = 90°$. Neglect the mass of the links.



Free - Body Diagram: When θ undergoes a positive virtual angular displacement of $\delta\theta$, the dash line configuration shown in Fig. *a* is formed. We observe that only the spring force $\mathbf{F}_{sp}$, the weight $\mathbf{W}_t$ of the table, and the weight $\mathbf{W}_b$ of the bowl do work when the virtual displacement takes place. The magnitude of $\mathbf{F}_{sp}$ can be computed using the spring force formula,

$F_{sp} = kx = k(0.25\cos\theta) = 0.25k\cos\theta.$

Virtual Displacement: The position of points of application of $\mathbf{W}_b$, $\mathbf{W}_t$, and $\mathbf{F}_{sp}$ are specified by the position coordinates y_{G_b}, y_{G_t}, and x_C, respectively. Here, y_{G_b} and y_{G_t} are measured from the fixed point B while x_C is measured from the fixed point D.

$$y_{G_b} = 0.25\sin\theta + b \qquad \delta y_{G_b} = 0.25\cos\theta\delta\theta \qquad (1)$$
$$y_{G_t} = 0.25\sin\theta + a \qquad \delta y_{G_t} = 0.25\cos\theta\delta\theta \qquad (2)$$
$$x_C = 0.25\cos\theta \qquad \delta x_C = -0.25\sin\theta\delta\theta \qquad (3)$$

Virtual Work Equation: Since $\mathbf{W}_b$, $\mathbf{W}_t$, and $\mathbf{F}_{sp}$ act towards the negative sense of their corresponding virtual displacement, their work is negative. Thus,

$$\delta U = 0; \qquad -W_b\delta y_{G_b} + \left(-W_t\delta y_{G_t}\right) + \left(-F_{sp}\delta x_C\right) = 0 \qquad (4)$$

Substituting $W_b = \left(\frac{1}{2}\right)(9.81) = 4.905\text{ N}$, $W_t = \left(\frac{5}{2}\right)(9.81) = 24.525\text{ N}$, $F_{sp} = 0.25k\cos\theta\text{N}$, Eqs. (1), (2), and (3) into Eq. (4), we have

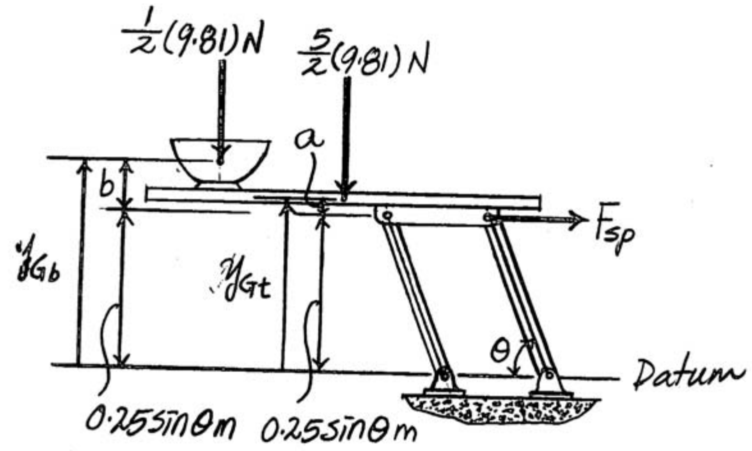
$$-4.905(0.25\cos\theta\delta\theta) - 24.525(0.25\cos\theta\delta\theta) - 0.25k\cos\theta(-0.25\sin\theta\delta\theta) = 0$$
$$\delta\theta(-7.3575\cos\theta + 0.0625k\sin\theta\cos\theta) = 0$$

Since $\delta\theta \neq 0$, then

$$-7.3575\cos\theta + 0.0625k\sin\theta\cos\theta = 0$$
$$k = \frac{117.72}{\sin\theta}$$

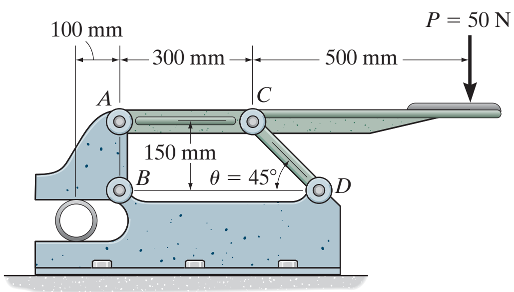
When $\theta = 45°$, then

$$k = \frac{117.72}{\sin 45°} = 166\text{ N/m} \qquad \textbf{Ans.}$$



(a)

11–18. If a vertical force of $P = 50$ N is applied to the handle of the toggle clamp, determine the clamping force exerted on the pipe.



Free - Body Diagram: When θ undergoes a positive virtual angular displacement of $\delta\theta$, the dash line configuration shown in Fig. *a* is formed. We observe that only force **P** and the clamping force **F** do work when the virtual displacement takes place.

Virtual Displacement: Since $\delta\theta$ is very small, the virtual displacement of point C can be appoximated by $\delta_C = \sqrt{0.045}\delta\theta$ m. From the geometry shown in Fig. *b*, the horizontal and vertical components of δ_C are given by $(\delta_C)_x = \delta_C \sin\theta = \sqrt{0.045}\sin\theta\delta\theta$ and $(\delta_C)_y = \delta_C \cos\theta = \sqrt{0.045}\cos\theta\delta\theta$, respectively. By referring to Fig. *a*, we notice that $\delta_A = (\delta_C)_x = \sqrt{0.045}\sin\theta\delta\theta$. Also,

$$\delta\phi = \frac{\delta_F}{0.1} = \frac{\delta_A}{0.15} \text{ or } \delta_F = 0.6667\delta_A = 0.6667\sqrt{0.045}\sin\theta\delta\theta$$

and

$$\delta x = \frac{(\delta_C)_y}{0.3} = \frac{\sqrt{0.045}\cos\theta\delta\theta}{0.3} = 3.333\sqrt{0.045}\cos\theta\delta\theta$$

and

$$\delta_P = 0.5\delta x + (\delta_C)_y = 0.5\left(3.333\sqrt{0.045}\cos\theta\delta\theta\right) + \sqrt{0.045}\cos\theta\delta\theta = 2.6667\sqrt{0.045}\cos\theta\delta\theta$$

Virtual Work Equation: Since **F** acts towards the positive sense of its corresponding virtual displacements, its work is positive. However, **P** does negative work since it acts in the negative sense of its corresponding virtual displacement. Thus,

$$\delta U = 0; \qquad F\delta_F + (-P\delta_P) = 0$$

Substituting $P = 50$ N and the results of δ_F and δ_P into the above equation

$$F(0.6667\sqrt{0.045}\sin\theta\delta\theta) - 50(2.6667\sqrt{0.045}\cos\theta\delta\theta) = 0$$

$$\sqrt{0.045}\delta\theta(0.6667F\sin\theta - 133.33\cos\theta) = 0$$

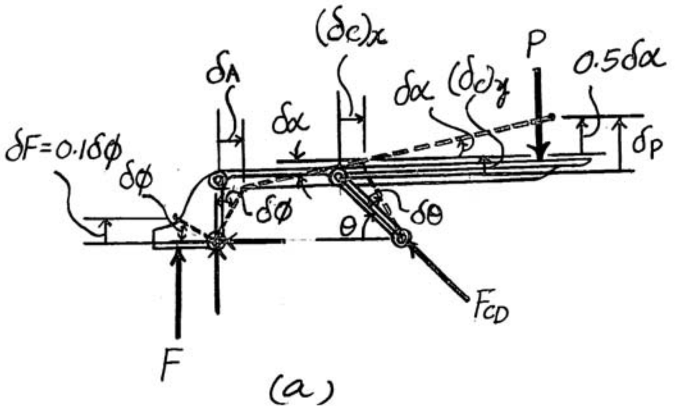
Since $\sqrt{0.045}\delta\theta \neq 0$, then

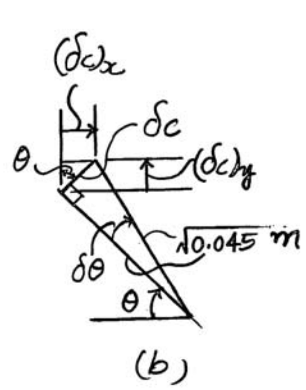
$$0.6667F\sin\theta - 133.33\cos\theta = 0$$

$$F = \frac{200\cos\theta}{\sin\theta}$$

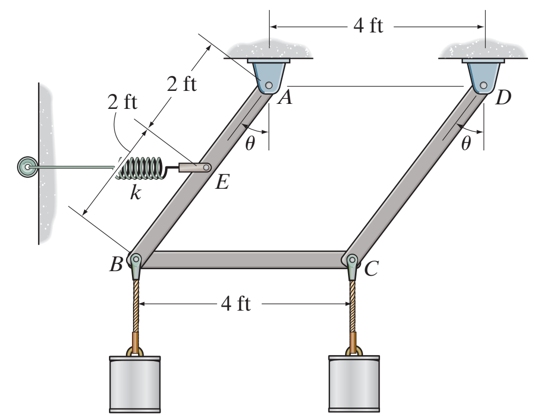
At $\theta = 45°$,

$$F = \frac{200\cos 45°}{\sin 45°} = 200\text{ N} \qquad \textbf{Ans.}$$





11–19. The spring is unstretched when $\theta = 45°$ and has a stiffness of $k = 1000$ lb/ft. Determine the angle θ for equilibrium if each of the cylinders weighs 50 lb. Neglect the weight of the members. The spring remains horizontal at all times due to the roller.



Free - Body Diagram: When θ undergoes a positive virtual angular displacement of $\delta\theta$, the dash line configuration shown in Fig. *a* is formed. We observe that only the spring force $\mathbf{F}_{sp}$ and the weight $\mathbf{W}$ of the cylinder do work when the virtual displacement takes place. The magnitude of $\mathbf{F}_{sp}$ can be computed using the spring force formula,

$$F_{sp} = kx = 1000(2\sin 45° - 2\sin\theta) = 2000(0.7071 - \sin\theta)\text{ lb}.$$

Virtual Displacement: The positions of the points of application of $\mathbf{W}$ and $\mathbf{F}_{sp}$ are specified by the position coordinates y_W and x_E, measured from the fixed point A.

$$y_W = 4\cos\theta + b \qquad \delta y_W = -4\sin\theta\delta\theta \qquad (1)$$

$$x_E = 2\sin\theta \qquad \delta x_E = 2\cos\theta\delta\theta \qquad (2)$$

Virtual Work Equation: Since $\mathbf{W}$ and $\mathbf{F}_{sp}$ act towards the positive sense of their corresponding virtual displacements, their work is positive. Thus,

$$\delta U = 0; \qquad 2W\delta y_W + F_{sp}\delta x_E = 0 \qquad (3)$$

Substituting $W = 50$ lb, $F_{sp} = 2000(0.7071 - \sin\theta)$, Eqs. (1), and (2) into Eq. (3),

$$2(50)(-4\sin\theta\delta\theta) + 2000(0.7071 - \sin\theta)(2\cos\theta\delta\theta) = 0$$

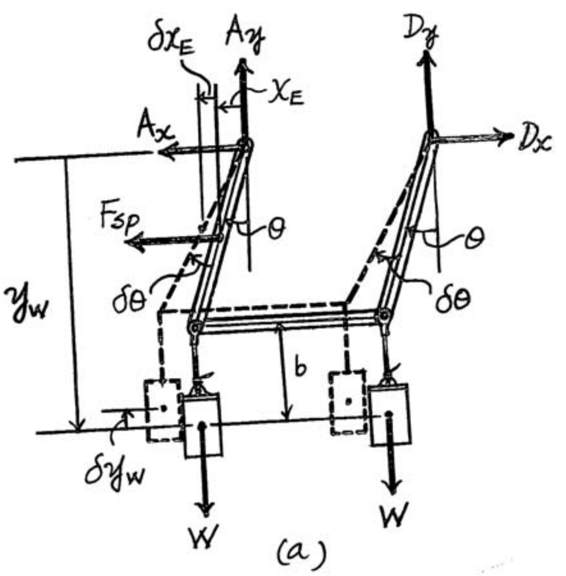
$$\delta\theta(-400\sin\theta + 2828.43\cos\theta - 4000\cos\theta\sin\theta) = 0$$

Since $\delta\theta \neq 0$, then

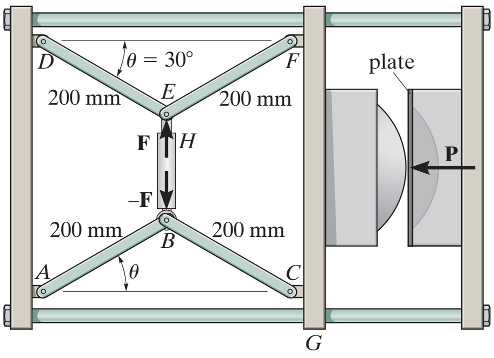
$$-400\sin\theta + 2828.43\cos\theta - 4000\cos\theta\sin\theta = 0$$

Solving by trial and error,

$$\theta = 38.8° \qquad \textbf{Ans.}$$



***11–20.** The machine shown is used for forming metal plates. It consists of two toggles ABC and DEF, which are operated by the hydraulic cylinder. The toggles push the moveable bar G forward, pressing the plate into the cavity. If the force which the plate exerts on the head is $P = 8$ kN, determine the force F in the hydraulic cylinder when $\theta = 30°$.



Free - Body Diagram: When θ undergoes a positive virtual angular displacement of $\delta\theta$, the dash line configuration shown in Fig. a is formed.

Virtual Displacement: The position of points of application of **F**, and **P** are specified by the position coordinates y_E, y_{B_t}, and x_G, respectively.

$$y_E = 0.2\sin\theta \qquad \delta y_E = 0.2\cos\theta\,\delta\theta \qquad (1)$$
$$y_B = 0.2\sin\theta \qquad \delta y_B = 0.2\cos\theta\,\delta\theta \qquad (2)$$
$$x_G = 2(0.2\cos\theta) \qquad \delta x_G = -0.4\sin\theta\,\delta\theta \qquad (3)$$

Virtual Work Equation: Thus,

$$\delta U = 0; \qquad -F\delta y_E + (-F\delta y_B) + (-P\delta x_G) = 0 \qquad (4)$$

Substituting Eqs. (1), (2), and (3) into Eq. (4), we have

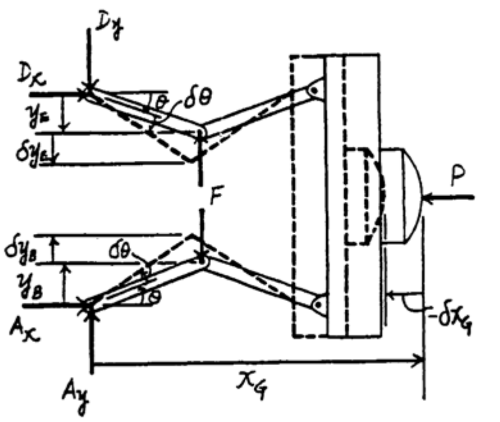
$$\delta\theta(-0.4F\cos\theta + 0.4P\sin\theta) = 0$$

Since $\delta\theta \neq 0$, then

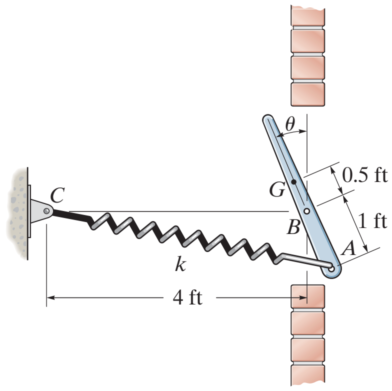
$$-0.4F\cos\theta + 0.4P\sin\theta = 0$$
$$F = P\tan\theta$$

When $\theta = 30°$, $P = 8$ kN then

$$F = 8\tan 30° = 4.62 \text{ kN} \qquad \textbf{Ans.}$$



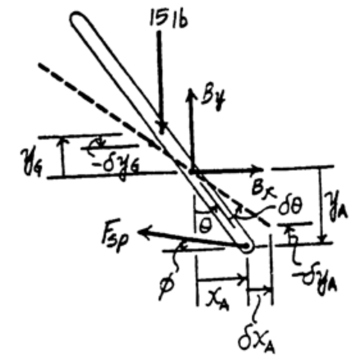
•11–21. The vent plate is supported at B by a pin. If it weighs 15 lb and has a center of gravity at G, determine the stiffness k of the spring so that the plate remains in equilibrium at $\theta = 30°$. The spring is unstretched when $\theta = 0°$.



Free Body Diagram : The system has only one degree of freedom defined by the independent coordinate θ. When θ undergoes a positive displacement $\delta\theta$, only the spring force F_{sp} and the weight of the vent plate (15 lb force) do work.

Virtual Displacements : The weight of the vent plate (15 lb force) is located from the fixed point B using the position coordinate y_G. The horizontal and vertical position of the spring force F_{sp} are measured from the fixed point B using the position coordinates x_A and y_A, respectively.

$$y_G = 0.5\cos\theta \qquad \delta y_G = -0.5\sin\theta\,\delta\theta \qquad [1]$$
$$y_A = 1\cos\theta \qquad \delta y_A = -\sin\theta\,\delta\theta \qquad [2]$$
$$x_A = 1\sin\theta \qquad \delta x_A = \cos\theta\,\delta\theta \qquad [3]$$



Virtual - Work Equation : When y_G, y_A and x_A undergo positive virtual displacements δy_G, δy_A and δx_A, the weight of the vent plate (15 lb force), horizontal component of F_{sp}, $F_{sp}\cos\phi$ and vertical component of F_{sp}, $F_{sp}\sin\phi$ do negative work.

$$\delta U = 0; \qquad -F_{sp}\cos\phi\,\delta x_A - F_{sp}\sin\phi\,\delta y_A - 15\delta y_G = 0 \qquad [4]$$

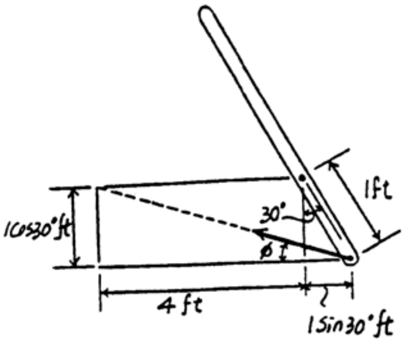
Substituting Eqs. [1], [2] and [3] into [4] yields

$$\left(-F_{sp}\cos\theta\cos\phi + F_{sp}\sin\theta\sin\phi + 7.5\sin\theta\right)\delta\theta = 0$$
$$\left(-F_{sp}\cos(\theta+\phi) + 7.5\sin\theta\right)\delta\theta = 0$$

Since $\delta\theta \neq 0$, then

$$-F_{sp}\cos(\theta+\phi) + 7.5\sin\theta = 0$$
$$F_{sp} = \frac{7.5\sin\theta}{\cos(\theta+\phi)}$$

At equilibrium position $\theta = 30°$, the angle $\phi = \tan^{-1}\left(\frac{1\cos 30°}{4 + 1\sin 30°}\right) = 10.89°$.

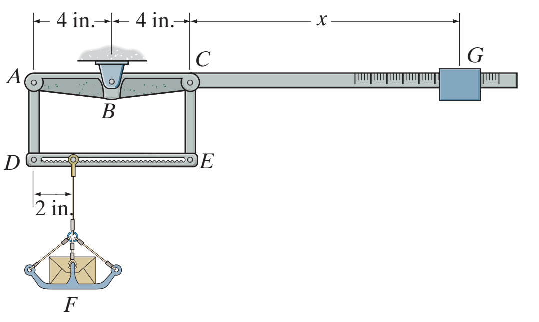


$$F_{sp} = \frac{7.5\sin 30°}{\cos(30° + 10.89°)} = 4.961 \text{ lb}$$

Spring Formula : From the geometry, the spring stretches
$x = \sqrt{4^2 + 1^2 - 2(4)(1)\cos 120°} - \sqrt{4^2 + 1^2} = 0.4595$ ft.

$$F_{sp} = kx$$
$$4.961 = k(0.4595)$$
$$k = 10.8 \text{ lb/ft} \qquad \textbf{Ans}$$

11–22. Determine the weight of block G required to balance the differential lever when the 20-lb load F is placed on the pan. The lever is in balance when the load and block are not on the lever. Take $x = 12$ in.



Free - Body Diagram: When the lever undergoes a virtual angular displacement of $\delta\theta$ about point B, the dash line configuration shown in Fig. *a* is formed. We observe that only the weight $\mathbf{W}_G$ of block G and the weight $\mathbf{W}_F$ of load F do work when the virtual displacements take place.

Virtual Displacement: Since δy_G is very small, the vertical virtual displacement of block G and load F can be approximated as

$$\delta y_G = (12+4)\delta\theta = 16\delta\theta \qquad (1)$$

$$\delta y_F = 2\delta\theta \qquad (2)$$

Virtual Work Equation: Since $\mathbf{W}_G$ acts towards the positive sense of its corresponding virtual displacement, its work is positive. However, force $\mathbf{W}_F$ does negative work since it acts towards the negative sense of its corresponding virtual displacement. Thus,

$$\delta U = 0; \qquad W_G \delta y_G + (-W_F \delta y_F) = 0 \qquad (3)$$

Substituting $W_G = 20$ lb and Eqs. (1) and (2) into Eq. (3),

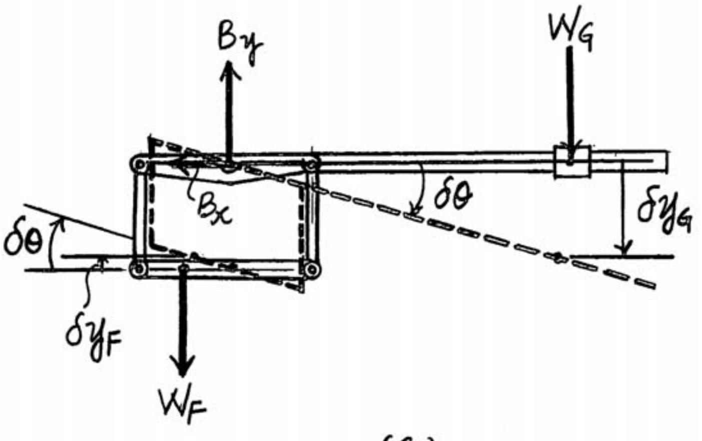
$$W_G(16\delta\theta) - 20(2\delta\theta) = 0$$

$$\delta\theta(16W_G - 40) = 0$$

Since $\delta\theta \neq 0$, then

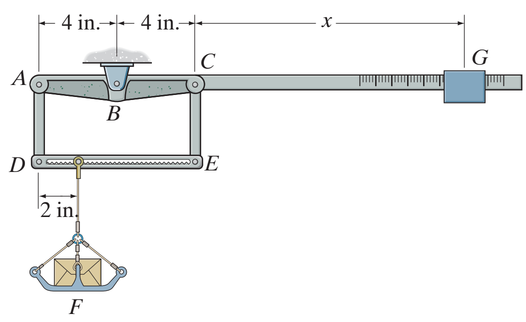
$$16W_G - 40 = 0$$

$$W_G = 2.5 \text{ lb} \qquad \textbf{Ans.}$$



(a)

11–23. If the load F weighs 20 lb and the block G weighs 2 lb, determine its position x for equilibrium of the differential lever. The lever is in balance when the load and block are not on the lever.



Free - Body Diagram: When the lever undergoes a virtual angular displacement of $\delta\theta$ about point B, the dash line configuration shown in Fig. a is formed. We observe that only the weight $\mathbf{W}_G$ of block G and the weight $\mathbf{W}_F$ of load F do work when the virtual displacements take place.

Virtual Displacement: Since δy_G is very small, the vertical virtual displacement of block G and load F can be approximated as

$$\delta y_G = (4+x)\delta\theta \qquad (1)$$

$$\delta y_F = 2\delta\theta \qquad (2)$$

Virtual Work Equation: Since $\mathbf{W}_G$ acts towards the positive sense of its corresponding virtual displacement, its work is positive. However, force $\mathbf{W}_F$ does negative work since it acts towards the negative sense of its corresponding virtual displacement. Thus,

$$\delta U = 0; \qquad W_G\delta y_G + (-W_F\delta y_F) = 0 \qquad (3)$$

Substituting $W_F = 20$ lb, $W_G = 2$ lb, Eqs. (1) and (2) into Eq. (3),

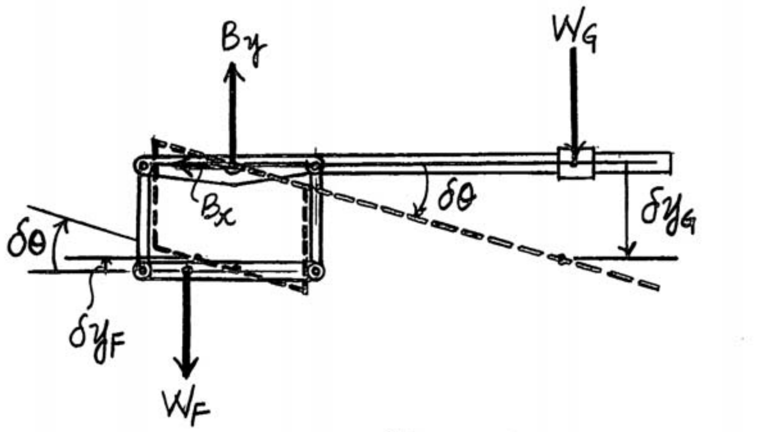
$$2(4+x)\delta\theta - 20(2\delta\theta) = 0$$

$$\delta\theta[2(4+x) - 40] = 0$$

Since $\delta\theta \neq 0$, then

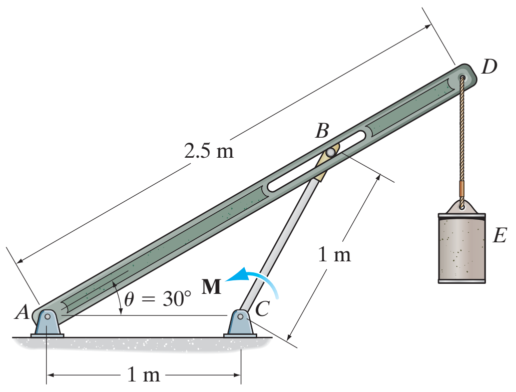
$$2(4+x) - 40 = 0$$

$$x = 16 \text{ in.} \qquad \textbf{Ans.}$$



(a)

***11–24.** Determine the magnitude of the couple moment **M** required to support the 20-kg cylinder in the configuration shown. The smooth peg at *B* can slide freely within the slot. Neglect the mass of the members.



Free - Body Diagram: When θ undergoes a positive virtual angular displacement of $\delta\theta$, the dash - line configuration shown in Fig. *a* is formed. We observe that only the couple moment **M** and the weight $\mathbf{W}_E$ of the cylinder do work when the virtual displacement takes place.

Virtual Displacement: The position of the point of application of $\mathbf{W}_E$ is specified by the position coordinate y_E, measured from the fixed point C.

$$y_E = b - 2.5\sin\theta \qquad \delta y_E = -2.5\cos\theta\,\delta\theta \qquad (1)$$

From the geometry shown in Fig. *b*, we obtain

$$\phi = 2\theta \qquad \delta\phi = 2\delta\theta \qquad (2)$$

Virtual Work Equation: Since **M** and $\mathbf{W}_E$ act in the positive sense of their corresponding virtual displacements, their work is positive. Thus,

$$\delta U = 0; \qquad M\delta\phi + W_E\,\delta y_E = 0 \qquad (3)$$

Substituting $W_E = 20(9.81)$ N and Eqs. (1) and (2) into Eq. (3),

$$M(2\delta\theta) + 20(9.81)(-2.5\cos\theta\,\delta\theta) = 0$$

$$\delta\theta(2M - 490.5\cos\theta) = 0$$

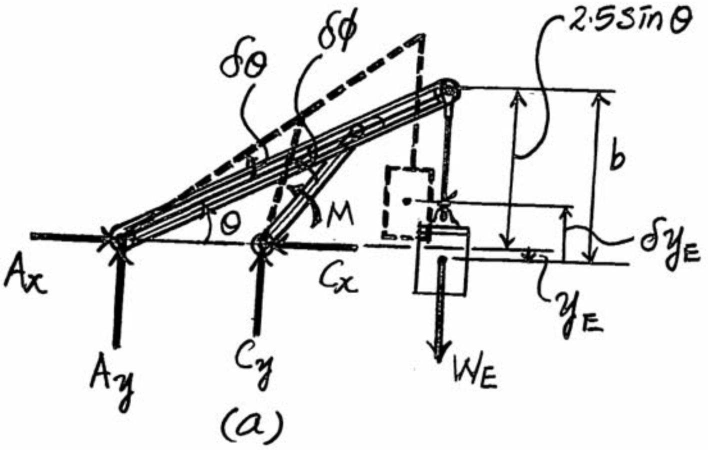
Since $\delta\theta \neq 0$, then

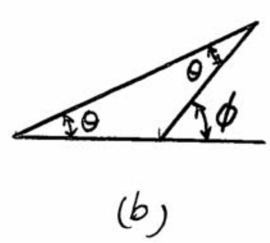
$$2M - 490.5\cos\theta = 0$$

$$M = 245.25\cos\theta$$

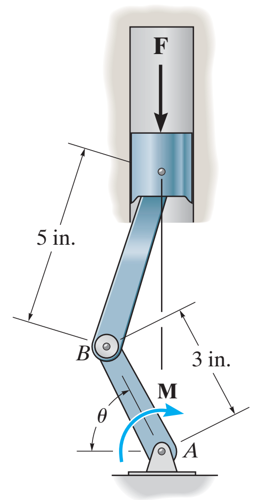
At $\theta = 30°$,

$$M = 245.25\cos 30° = 212\ \text{N}\cdot\text{m} \qquad \textbf{Ans.}$$





•**11–25.** The crankshaft is subjected to a torque of $M = 50\ \text{lb}\cdot\text{ft}$. Determine the vertical compressive force **F** applied to the piston for equilibrium when $\theta = 60°$.



Free Body Diagram : The system has only one degree of freedom defined by the independent coordinate θ. When θ undergoes a positive displacement $\delta\theta$, only the force **F** and couple moment **M** do work.

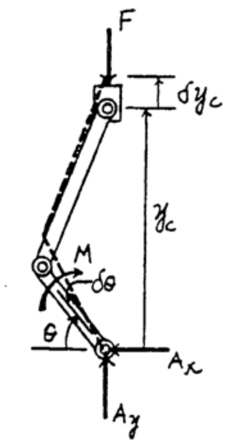
Virtual Displacements : Force **F** is located from the fixed point A using the positional coordinate y_C. Using the law of cosines.

$$5^2 = y_C^2 + 3^2 - 2(y_C)(3)\cos(90° - \theta) \qquad [1]$$

However, $\cos(90° - \theta) = \sin\theta$. Then Eq.[1] becomes $25 = y_C^2 + 9 - 6y_C\sin\theta$. Differentiating this expression, we have

$$0 = 2y_C\delta y_C - 6\delta y_C\sin\theta - 6y_C\cos\theta\delta\theta$$

$$\delta y_C = \frac{6y_C\cos\theta}{2y_C - 6\sin\theta}\delta\theta \qquad [2]$$



Virtual-Work Equation : When point C undergoes a positive virtual displacement δy_C, force **F** does negative work. The couple moment **M** does positive work when link AB undergoes a positive virtual rotation $\delta\theta$.

$$\delta U = 0; \qquad -F\delta y_C + M\delta\theta = 0 \qquad [3]$$

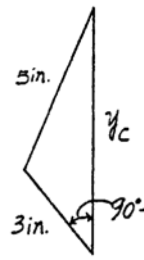
Substituting Eq.[2] into [3] yields

$$\left(-\frac{6y_C\cos\theta}{2y_C - 6\sin\theta}F + M\right)\delta\theta = 0$$

Since $\delta\theta \neq 0$, then

$$-\frac{6y_C\cos\theta}{2y_C - 6\sin\theta}F + M = 0$$

$$F = \frac{2y_C - 6\sin\theta}{6y_C\cos\theta}M \qquad [4]$$



At the equilibrium position, $\theta = 60°$. Substituting into Eq.[1], we have

$$5^2 = y_C^2 + 3^2 - 2(y_C)(3)\cos 30°$$

$$y_C = 7.368\ \text{in.}$$

Substituting the above results into Eq. [4] and setting $M = 50\ \text{lb}\cdot\text{ft}$, we have

$$F = \left[\frac{2(7.368) - 6\sin 60°}{6(7.368)\cos 60°}\right]50(12\ \text{in.})/\text{ft} = 259\ \text{lb} \qquad \textbf{Ans.}$$

11–26. If the potential energy for a conservative one-degree-of-freedom system is expressed by the relation $V = (4x^3 - x^2 - 3x + 10)$ ft·lb, where x is given in feet, determine the equilibrium positions and investigate the stability at each position.

$$V = 4x^3 - x^2 - 3x + 10$$

Equilibrium Position :

$$\frac{dV}{dx} = 12x^2 - 2x - 3 = 0$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(12)(-3)}}{24}$$

$x = 0.590$ ft and -0.424 ft **Ans**

Stability :

$$\frac{d^2V}{dx^2} = 24x - 2$$

At $x = 0.590$ ft $\quad \frac{d^2V}{dx^2} = 24(0.590) - 2 = 12.2 > 0$ **stable** **Ans**

At $x = -0.424$ ft $\quad \frac{d^2V}{dx^2} = 24(-0.424) - 2 = -12.2 < 0$ **unstable** **Ans**

11–27. If the potential energy for a conservative one-degree-of-freedom system is expressed by the relation $V = (24 \sin\theta + 10 \cos 2\theta)$ ft·lb, $0° \le \theta \le 90°$, determine the equilibrium positions and investigate the stability at each position.

$$V = 24\sin\theta + 10\cos 2\theta$$

Equilibrium Position :

$$\frac{dV}{d\theta} = 24\cos\theta - 20\sin 2\theta = 0$$

$$24\cos\theta - 40\sin\theta\cos\theta = 0$$

$$\cos\theta(24 - 40\sin\theta) = 0$$

$\cos\theta = 0 \qquad \theta = 90°$ **Ans**

$24 - 40\sin\theta = 0 \qquad \theta = 36.9°$ **Ans**

Stability :

$$\frac{d^2V}{d\theta^2} = -40\cos 2\theta - 24\sin\theta$$

At $\theta = 90°$ $\quad \frac{d^2V}{d\theta^2} = -40\cos 180° - 24\sin 90° = 16 > 0$ **stable** **Ans**

At $\theta = 36.9°$ $\quad \frac{d^2V}{d\theta^2} = -40\cos 73.7° - 24\sin 36.9° = -25.6 < 0$ **unstable** **Ans**

***11–28.** If the potential energy for a conservative one-degree-of-freedom system is expressed by the relation $V = (3y^3 + 2y^2 - 4y + 50)$ J, where y is given in meters, determine the equilibrium positions and investigate the stability at each position.

Potential Function:

$$V = 3y^3 + 2y^2 - 4y + 50$$

Equilibrium Configuration: Taking the first derivative of V,

$$\frac{dV}{d\theta} = 9y^2 + 4y - 4 = 0$$

Thus,

$y = 0.481$ m, $y = -925$ m **Ans.**

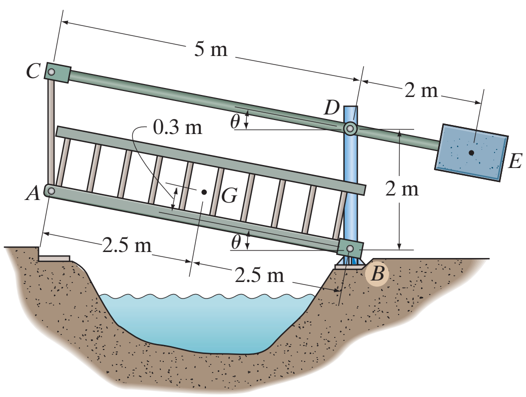
Stability: The second derivative of V is

$$\frac{d^2V}{d\theta^2} = 18y + 4$$

At $y = 0.481$ m, $\frac{d^2V}{d\theta^2} = 12.7 > 0$ Stable **Ans.**

At $y = -925$ m, $\frac{d^2V}{d\theta^2} = -12.7 < 0$ Unstable **Ans.**

•**11–29.** The 2-Mg bridge, with center of mass at point G, is lifted by two beams CD, located at each side of the bridge. If the 2-Mg counterweight E is attached to the beams as shown, determine the angle θ for equilibrium. Neglect the weight of the beams and the tie rods.



Potential Function: With reference to the datum, Fig. *a*, the gravitational potential energy of the bridge and counterweight are positive since their centers of gravity are located above the datum. Referring to the geometry shown in Fig. *b*,

$y_G = (0.3\cos\theta + 2.5\sin\theta)$ m and $y_E = (2 - 2\sin\theta)$ m

Thus,

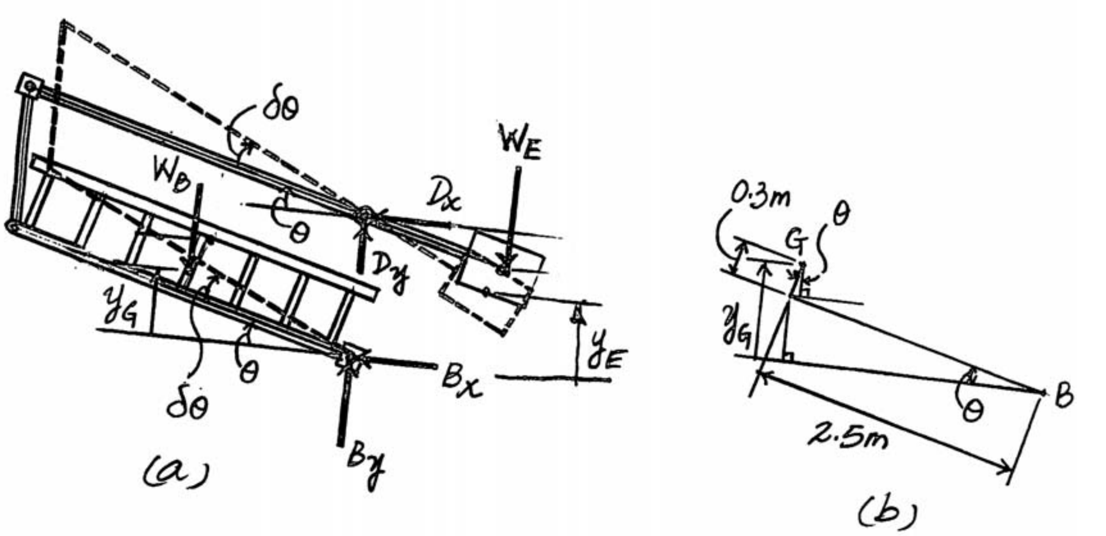
$$V = V_g = \Sigma mgy = 2000(9.81)(0.3\cos\theta + 2.5\sin\theta) + 2000(9.81)(2 - 2\sin\theta)$$
$$= 5886\cos\theta + 9810\sin\theta + 39240$$

Equilibrium Configuration: Taking the first derivative of V,

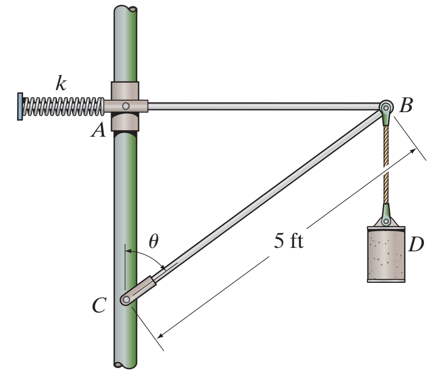
$$\frac{dV}{d\theta} = -5886\sin\theta + 9810\cos\theta$$

Equilibrium requires $\frac{dV}{d\theta} = 0$. Thus,

$$-5886\sin\theta + 9810\cos\theta = 0$$
$$\theta = 59.04° = 59.0° \qquad \textbf{Ans.}$$



11–30. The spring has a stiffness $k = 600$ lb/ft and is unstretched when $\theta = 45°$. If the mechanism is in equilibrium when $\theta = 60°$, determine the weight of cylinder D. Neglect the weight of the members. Rod AB remains horizontal at all times since the collar can slide freely along the vertical guide.



Potential Function: With reference to the datum, Fig. *a*, the gravitational potential energy of the cylinder is positive since its center of gravity is located above the datum. Here, $y = (5\cos\theta - b)$ ft. Thus,

$$V_g = Wy = W_D(5\cos\theta - b) = 5W_D\cos\theta - W_D b$$

The elastic potential energy of the spring can be computed using $V_g = \frac{1}{2}ks^2$, where $s = (5\sin\theta - 5\sin 45°)$ ft $= (5\sin\theta - 3.5355)$ ft.

Thus,

$$V_e = \frac{1}{2}(600)(5\sin\theta - 3.5355)^2 = 7500\sin^2\theta - 10606.60\sin\theta + 3750$$

The total potential energy of the system is

$$V = V_g + V_e = 5W_D\cos\theta + 7500\sin^2\theta - 10606.60\sin\theta - W_D b + 3750$$

Equilibrium Configuration: Taking the first derivative of V,

$$\frac{dV}{d\theta} = -5W_D\sin\theta + 15000\sin\theta\cos\theta - 10606.60\cos\theta$$

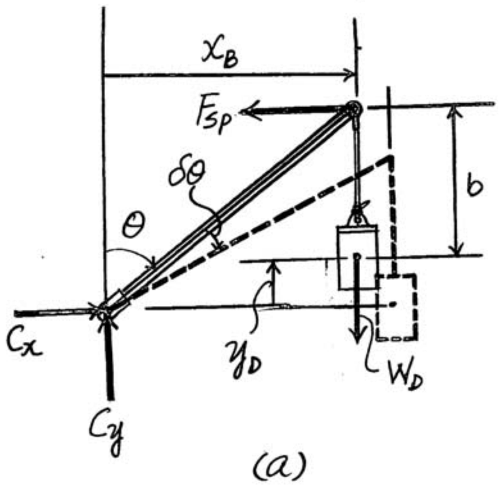
Equilibrium requires $\frac{dV}{d\theta} = 0$. Thus,

$$-5W_D\sin\theta + 15000\sin\theta\cos\theta - 10606.60\cos\theta = 0$$

$$W_D = \frac{15000\sin\theta\cos\theta - 10606.60\cos\theta}{5\sin\theta}$$

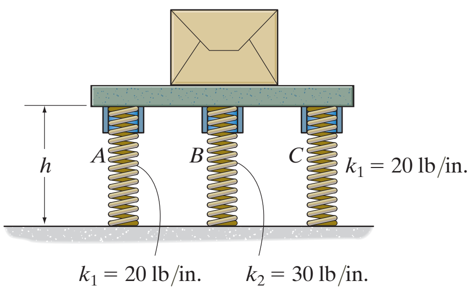
When $\theta = 60°$,

$$W_D = \frac{15000\sin 60°\cos 60° - 10606.60\cos 60°}{5\sin 60°} = 275 \text{ lb}$$ **Ans.**



(a)

11–31. If the springs at A and C have an unstretched length of 10 in. while the spring at B has an unstretched length of 12 in., determine the height h of the platform when the system is in equilibrium. Investigate the stability of this equilibrium configuration. The package and the platform have a total weight of 150 lb.



Potential Function: With reference to the datum, Fig. a, the gravitational potential energy of the package and the platform is positive since their center of gravity is located above the datum. Here, $y = h + b$ where b is a constant. Thus,

$$V_g = Wy = 150(h+b) = 150h + 150b$$

The elastic potential energy for the springs can be computed using $V_e = \frac{1}{2}ks^2$. Here, the compressions of the springs are $s_A = s_C = (10 - h)$ in. and $s_B = (12 - h)$ in. Thus,

$$V_e = 2\left[\frac{1}{2}(20)(10-h)^2\right] + \frac{1}{2}(30)(12-h)^2$$

$$= 35h^2 - 760h + 4160$$

The total potential energy of the system is

$$V = V_g + V_e = 35h^2 - 760h + 4160 + 150h + 150b$$

Equilibrium Configuration: Taking the first derivative of V,

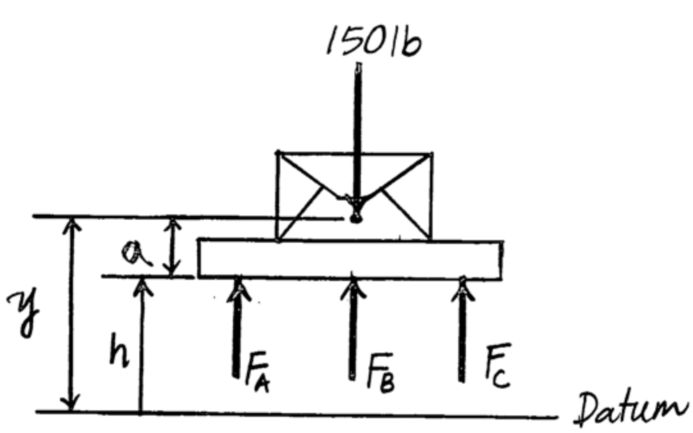
$$\frac{dV}{dh} = 70h - 610$$

Equilibrium requires $\frac{dV}{dh} = 0$. Thus,

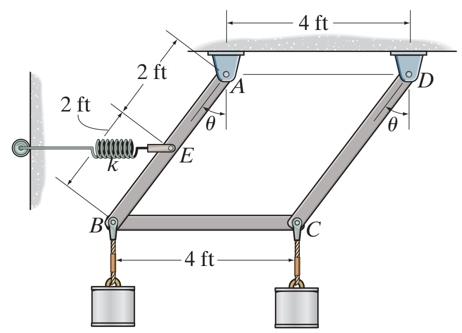
$$70h - 610 = 0 \qquad h = 8.714 \text{ in.} = 8.71 \text{ in.} \qquad \textbf{Ans.}$$

Stability: The second derivative of V is

$$\frac{d^2V}{dh^2} = 70 > 0 \qquad \textit{stable} \qquad \textbf{Ans.}$$



***11–32.** The spring is unstretched when $\theta = 45°$ and has a stiffness of $k = 1000$ lb/ft. Determine the angle θ for equilibrium if each of the cylinders weighs 50 lb. Neglect the weight of the members.



Potential Function: With reference to the datum, Fig. *a*, the gravitational potential energy of the cylinders is negative since their centers of gravity are located below the datum. Here, $y = 4\cos\theta + b$, where b is constant. Thus,

$$V_g = \Sigma Wy = -2\left[50(4\cos\theta + b)\right] = -(400\cos\theta + 100b)$$

The elastic potential energy of the spring can be computed using $V_g = \frac{1}{2}ks^2$, where $s = 2\sin 45° - 2\sin\theta = 1.414 - 2\sin\theta$. Thus,

$$V_e = \frac{1}{2}(1000)(1.414 - 2\sin\theta)^2 = 2000\sin^2\theta - 2828.43\sin\theta + 1000$$

The total potential energy of the system is

$$V = V_g + V_e = 2000\sin^2\theta - 2828.43\sin\theta - 400\cos\theta - 100b + 1000$$

Equilibrium Configuration: Taking the first derivative of V,

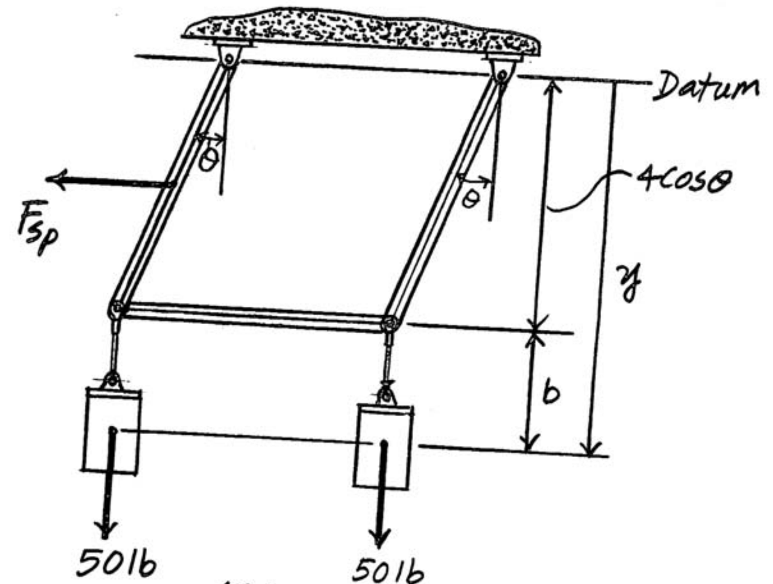
$$\frac{dV}{d\theta} = 4000\sin\theta\cos\theta - 2828.43\cos\theta + 400\sin\theta$$

Equilibrium requires $\frac{dV}{d\theta} = 0$. Thus,

$$4000\sin\theta\cos\theta - 2828.43\cos\theta + 400\sin\theta = 0$$

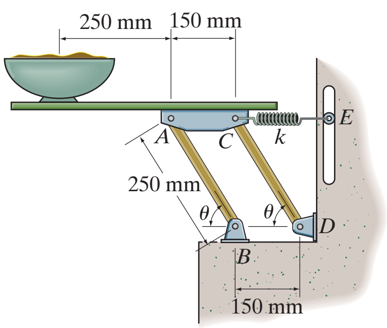
Solving by trial and error,

$$\theta = 38.8° \qquad \textbf{Ans.}$$



(a)

•11–33. A 5-kg uniform serving table is supported on each side by pairs of two identical links, AB and CD, and springs CE. If the bowl has a mass of 1 kg, determine the angle θ where the table is in equilibrium. The springs each have a stiffness of $k = 200$ N/m and are unstretched when $\theta = 90°$. Neglect the mass of the links.



Potential Function: With reference to the datum, Fig. a, the gravitational potential energy of the bowl and the table are positive since their centers of gravity are located above the datum. Here, $y_{Gt} = (0.25\sin\theta + a)$ m and $y_{Gb} = (0.25\sin\theta + b)$ m. Thus,

$$V_g = \Sigma mgy = \frac{5}{2}(9.81)(0.25\sin\theta + a) + \frac{1}{2}(9.81)(0.25\sin\theta + b)$$
$$= 7.3575\sin\theta + 24.525a + 4.905b$$

The elastic potential energy of the spring can be computed using $V_e = \frac{1}{2}ks^2$, where $s = 0.25\cos\theta$ m. Thus,

$$V_e = \frac{1}{2}(200)(0.25\cos\theta)^2 = 6.25\cos^2\theta$$

The total potential energy of the system is

$$V = V_g + V_e = 6.25\cos^2\theta + 7.3575\sin\theta + 24.525a + 4.905b$$

Equilibrium Configuration: Taking the first derivative of V,

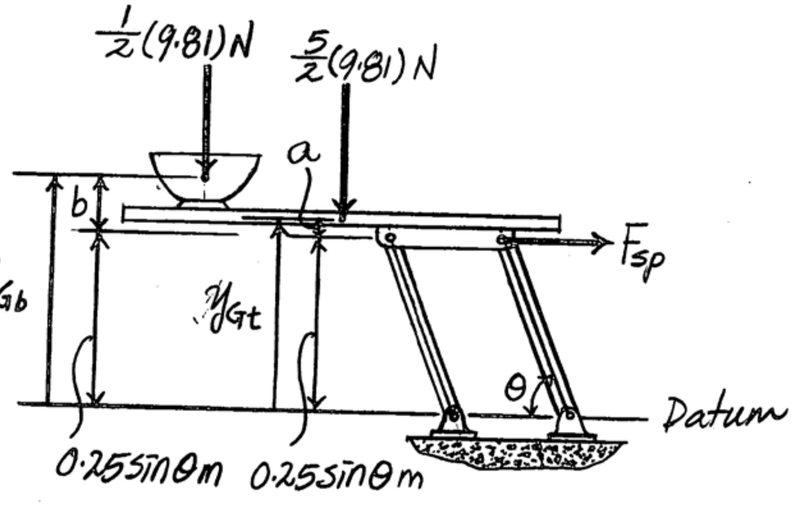
$$\frac{dV}{d\theta} = -12.5\cos\theta\sin\theta + 7.3575\cos\theta$$

Equilibrium requires $\frac{dV}{d\theta} = 0$. Thus,

$$-12.5\cos\theta\sin\theta + 7.3575\cos\theta = 0$$
$$\cos\theta(-12.5\sin\theta + 7.3575) = 0$$

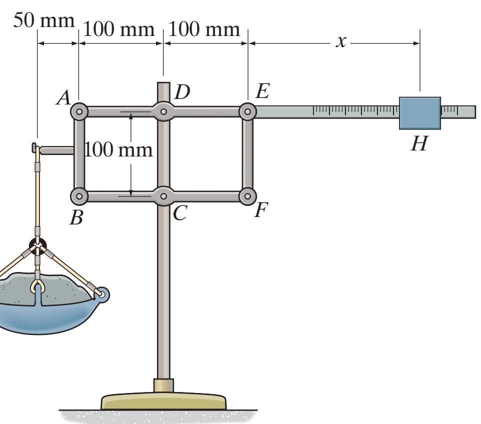
$\cos\theta = 0 \quad \theta = 90°$ **Ans.**

$-12.5\sin\theta + 7.3575 = 0 \qquad \theta = 36.1°$ **Ans.**



(a)

11–34. If a 10-kg load I is placed on the pan, determine the position x of the 0.75-kg block H for equilibrium. The scale is in balance when the weight and the load are not on the scale.



Potential Function: With reference to the datum, Fig. a, the gravitational potential energy of block H is positive since its center of gravity is located above the datum, while the gravitational potential energy of load I is negative since its center of gravity is located below the datum. Here, $y_H = [(0.1+x)\sin\theta]$ m and $y_I = (0.1\sin\theta + b)$ m where b is a constant. Thus,

$$V = V_g = \Sigma mgy = 0.75(9.81)(0.1+x)\sin\theta + [-10(9.81)(0.1\sin\theta + b)]$$
$$= 7.3575(0.1+x)\sin\theta - 9.81\sin\theta - 98.1b$$

Equilibrium Configuration: Taking the first derivative of V,

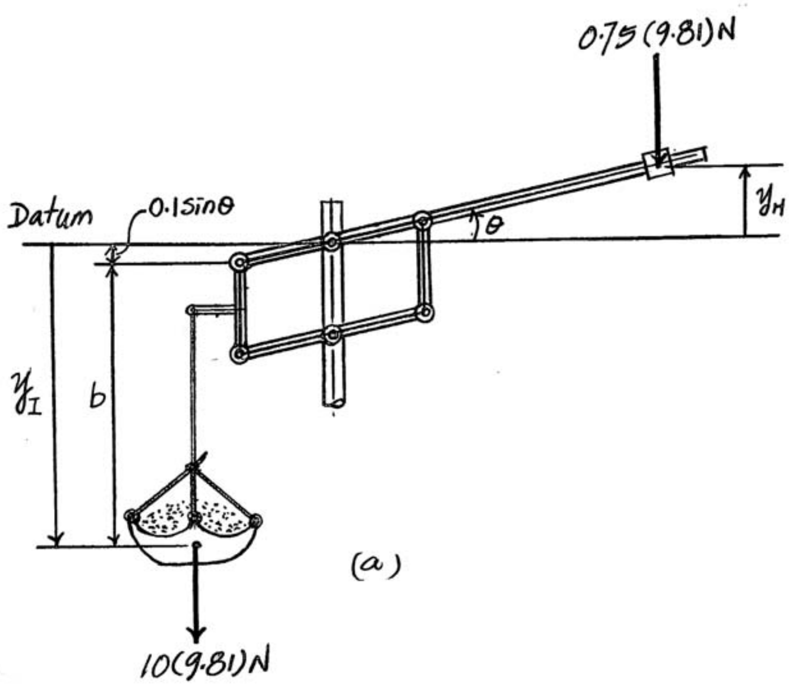
$$\frac{dV}{d\theta} = 7.3575(0.1+x)\cos\theta - 9.81\cos\theta$$

Equilibrium requires $\frac{dV}{d\theta} = 0$. Thus,

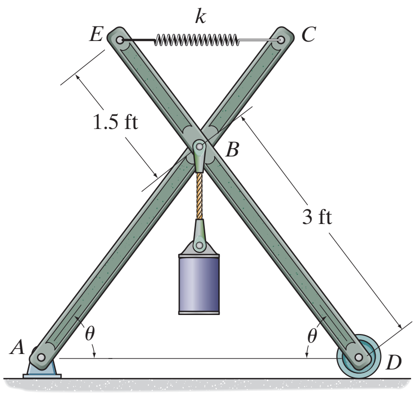
$$7.3575(0.1+x)\cos\theta - 9.81\cos\theta = 0$$
$$\cos\theta[7.3575(0.1+x) - 9.81] = 0$$
$$\cos\theta = 0 \quad \theta = 90°$$

or

$$7.3575(0.1+x) - 9.81 = 0$$
$$x = 1.23\text{ m}$$ **Ans.**



11–35. Determine the angles θ for equilibrium of the 200-lb cylinder and investigate the stability of each position. The spring has a stiffness of $k = 300$ lb/ft and an unstretched length of 0.75 ft.



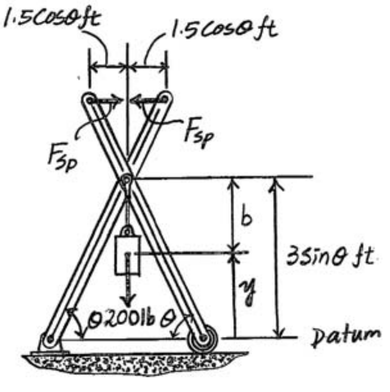
Potential Function: With reference to the datum, Fig. *a*, the gravitational potential energy of the cylinder is positive since its center of gravity is located above the datum. Here, $y = (3\sin\theta - b)$ ft, where b is a constant. Thus,

$$V_g = Wy = 200(3\sin\theta - b) = 600\sin\theta - 200b$$

The elastic potential energy of spring BC can be computed using $V_e = \frac{1}{2}ks^2$, where $s = 2(1.5\cos\theta) - 0.75 = (3\cos\theta - 0.75)$ ft. Thus,

$$V_e = \frac{1}{2}(300)(3\cos\theta - 0.75)^2$$

$$= 1350\cos^2\theta - 675\cos\theta + 84.75$$



(a)

The total potential energy of the system is

$$V = V_g + V_e = 1350\cos^2\theta - 675\cos\theta + 600\sin\theta + 84.375 - 200b$$

Equilibrium Configuration: Taking the first derivative of V,

$$\frac{dV}{d\theta} = -2700\cos\theta\,\sin\theta + 675\sin\theta + 600\cos\theta$$

$$= -1375\sin 2\theta + 675\sin\theta + 600\cos\theta$$

Equilibrium requires $\frac{dV}{d\theta} = 0$. Thus,

$$-1375\sin 2\theta + 675\sin\theta + 600\cos\theta = 0$$

Solving by trial and error,

$\theta = 17.1°$ and $\theta = 70.9°$ **Ans.**

Stability: The second derivative of V is

$$\frac{d^2V}{d^2\theta} = -2700\cos 2\theta + 675\cos\theta - 600\sin\theta$$

At the equilibrium configuration $\theta = 17.1°$,

$$\left.\frac{d^2V}{d^2\theta}\right|_{\theta=17.1°} = -2700\cos 34.2° + 675\cos 17.1° - 600\sin 17.1°$$

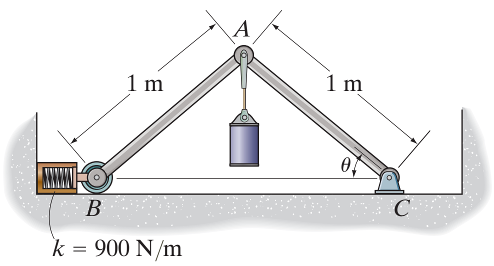
$= -1764 < 0$ *unstable* **Ans.**

At the equilibrium configuration $\theta = 70.92°$,

$$\left.\frac{d^2V}{d^2\theta}\right|_{\theta=70.92°} = -2700\cos 141.84° + 675\cos 70.92° - 600\sin 79.2°$$

$= 1776.67 > 0$ *stable* **Ans.**

***11–36.** Determine the angles θ for equilibrium of the 50-kg cylinder and investigate the stability of each position. The spring is uncompressed when $\theta = 60°$.



Potential Function: With reference to the datum, Fig. a, the gravitational potential energy of the cylinder is positive since its center of gravity is located above the datum. Here, $y = (1\sin\theta - b)$ m, where b is a constant. Thus,

$$V_g = mgy = 50(9.81)(\sin\theta - b) = 490.5\sin\theta - 490.5b$$

The elastic potential energy of the spring can be computed using $V_e = \frac{1}{2}ks^2$, where $s = 2(1\cos\theta - \cos 60°) = (2\cos\theta - 1)$ m. Thus,

$$V_e = \frac{1}{2}(900)(2\cos\theta - 1)^2 = 1800\cos^2\theta - 1800\cos\theta + 450$$

The total potential energy of the system is

$$V = V_g + V_e = 1800\cos^2\theta - 1800\cos\theta + 490.5\sin\theta + 450 - 490.5b$$

Equilibrium Configuration: Taking the first derivative of V,

$$\frac{dV}{d\theta} = -3600\sin\theta\cos\theta + 1800\sin\theta + 490.5\cos\theta$$
$$= -1800\sin 2\theta + 1800\sin\theta + 490.5\cos\theta$$

Equilibrium requires $\frac{dV}{d\theta} = 0$. Thus,

$$-1800\sin 2\theta + 1800\sin\theta + 490.5\cos\theta = 0$$

Solving by trial and error,

$$\theta = 16.55 = 16.6° \quad \text{and } \theta = 52.9°$$ **Ans.**

Stability: The second derivative of V is

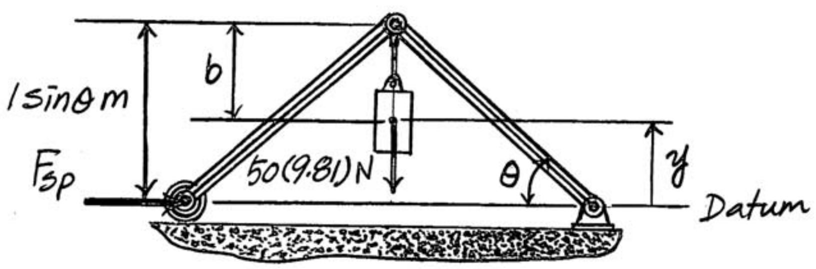
$$\frac{d^2V}{d^2\theta} = -3600\cos 2\theta + 1800\cos\theta - 490.5\sin\theta$$

At the equilibrium configuration $\theta = 16.55°$,

$$\left.\frac{d^2V}{d^2\theta}\right|_{\theta=52.916°} = -3600\cos 33.10° + 1800\cos 16.55° - 490.5\sin 16.55°$$
$$= -1430 < 0 \quad \textit{unstable}$$ **Ans.**

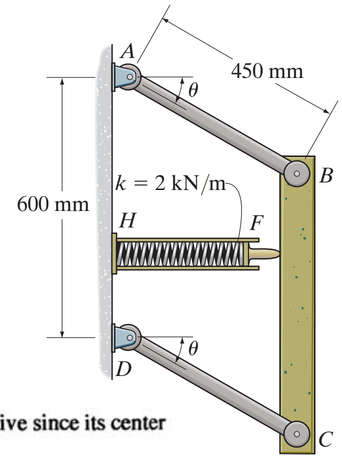
At the equilibrium configuration $\theta = 52.92°$,

$$\left.\frac{d^2V}{d^2\theta}\right|_{\theta=52.916°} = -3600\cos 105.84° + 1800\cos 52.92° - 490.5\sin 52.92°$$
$$= 1676.22 > 0 \quad \textit{stable}$$ **Ans.**



(a)

•11–37. If the mechanism is in equilibrium when $\theta = 30°$, determine the mass of the bar BC. The spring has a stiffness of $k = 2$ kN/m and is uncompressed when $\theta = 0°$. Neglect the mass of the links.



Potential Function: With reference to the datum, Fig. a, the gravitational potential energy of block E is negative since its center of gravity is located below the datum. Here, $y = (0.45\sin\theta + b)$ m, where b is a constant.

$$V_g = -mgy = -m_E(9.81)(0.45\sin\theta + b) = -(4.4145m_E\sin\theta + 9.81m_Eb)$$

The elastic potential energy of the spring can be computed using $V_e = \frac{1}{2}ks^2$, where $s = 0.45 - 0.45\cos\theta$. Thus,

$$V_e = \frac{1}{2}(2000)(0.45 - 0.45\cos\theta)^2$$
$$= 202.5 + 202.5\cos^2\theta - 405\cos\theta$$

The total potential energy of the system is

$$V = V_g + V_e = -4.4145m_E\sin\theta + 202.5\cos^2\theta - 405\cos\theta - 9.81m_Eb + 202.5$$

Equilibrium Configuration: Taking the first derivative of V,

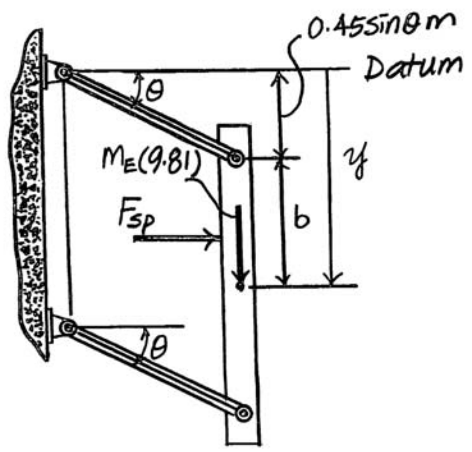
$$\frac{dV}{d\theta} = -4.4145m_E\cos\theta - 405\cos\theta\sin\theta + 405\sin\theta$$

Equilibrium requires $\frac{dV}{d\theta} = 0$. Thus,

$$-4.4145m_E\cos\theta - 405\cos\theta\sin\theta + 405\sin\theta = 0$$
$$m_E = \frac{405\sin\theta - 405\cos\theta\sin\theta}{4.4145\cos\theta}$$

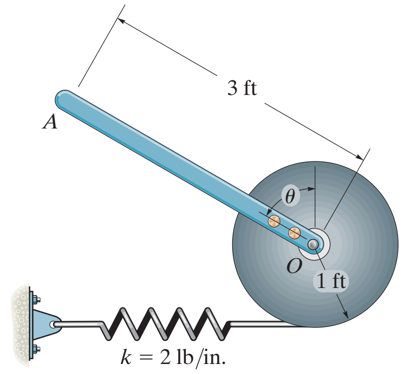
When $\theta = 30°$,

$$m_E = \frac{405\sin 30° - 405\cos 30°\sin 30°}{4.4145\cos 30°} = 7.10\text{ kg}$$ **Ans.**



(a)

11–38. The uniform rod *OA* weighs 20 lb, and when the rod is in the vertical position, the spring is unstretched. Determine the position θ for equilibrium. Investigate the stability at the equilibrium position.



Potential Function : The spring stretches $s = 12(\theta)$ in., where θ is in radians.

$$V = V_e + V_g = \frac{1}{2}(2)(12\theta)^2 + 20[1.5(12)\cos\theta]$$

$$= 144\theta^2 + 360\cos\theta$$

Equilibrium Position : $\dfrac{dV}{d\theta} = 0$

$$\frac{dV}{d\theta} = 288\theta - 360\sin\theta = 0$$

$$\theta = 1.1311 \text{ rad} = 64.8^\circ \qquad \textbf{Ans}$$

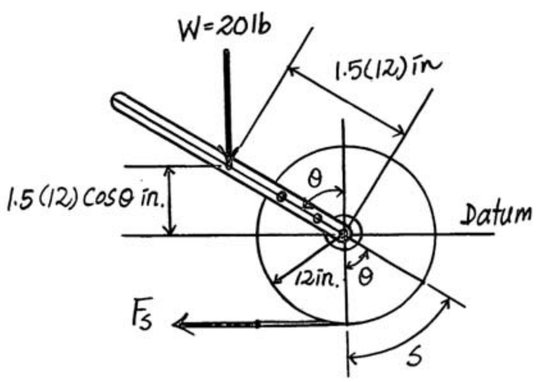
$$\theta = 0^\circ \qquad \textbf{Ans}$$

Stability :

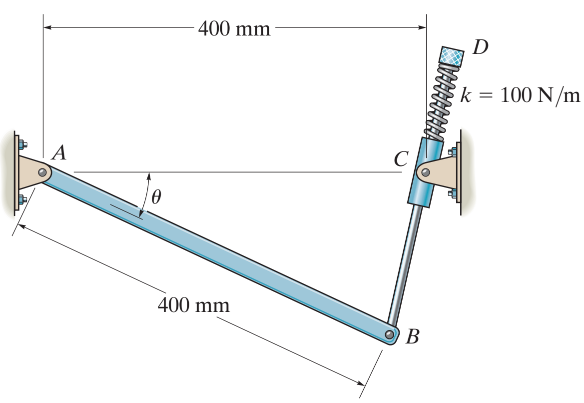
$$\frac{d^2V}{d\theta^2} = 288 - 360\cos\theta$$

At $\theta = 64.8^\circ$, $\dfrac{d^2V}{d\theta^2} = 288 - 360\cos 64.8^\circ = 135 > 0$ **stable** **Ans**

At $\theta = 0^\circ$, $\dfrac{d^2V}{d\theta^2} = 288 - 360\cos 0^\circ = -72 < 0$ **unstable** **Ans**



11–39. The uniform link *AB* has a mass of 3 kg and is pin connected at both of its ends. The rod *BD*, having negligible weight, passes through a swivel block at *C*. If the spring has a stiffness of $k = 100$ N/m and is unstretched when $\theta = 0^\circ$, determine the angle θ for equilibrium and investigate the stability at the equilibrium position. Neglect the size of the swivel block.



$$s = \sqrt{(0.4)^2 + (0.4)^2 - 2(0.4)^2\cos\theta}$$

$$= (0.4)\sqrt{2(1 - \cos\theta)}$$

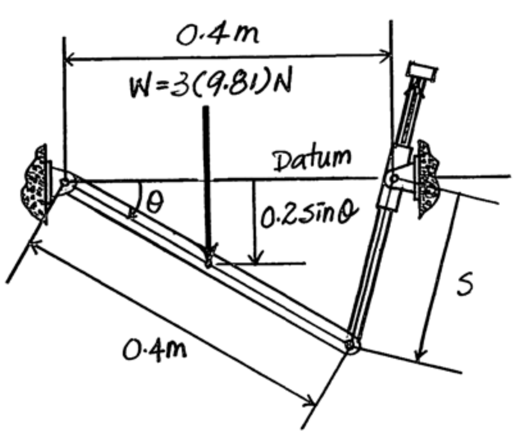
$$V = V_g + V_e$$

$$= -(0.2)(\sin\theta)3(9.81) + \frac{1}{2}(100)[(0.4)^2(2)(1 - \cos\theta)]$$

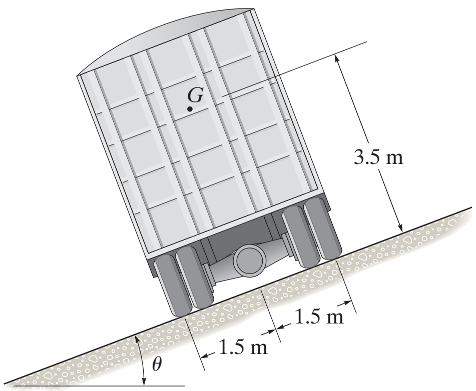
$$\frac{dV}{d\theta} = -(5.886)\cos\theta + 16(\sin\theta) = 0 \qquad (1)$$

$$\theta = 20.2^\circ \qquad \textbf{Ans}$$

$$\frac{d^2V}{d\theta^2} = 5.886\sin\theta + (16)\cos\theta = 17.0 > 0 \qquad \textbf{stable} \qquad \textbf{Ans}$$



***11–40.** The truck has a mass of 20 Mg and a mass center at G. Determine the steepest grade θ along which it can park without overturning and investigate the stability in this position.



Potential Function : The datum is established at point A. Since the center of gravity for the truck is above the datum, its potential energy is positive. Here, $y = (1.5\sin\theta + 3.5\cos\theta)$ m.

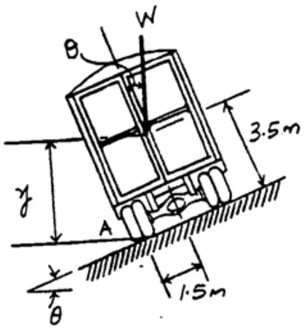
$$V = V_g = Wy = W(1.5\sin\theta + 3.5\cos\theta)$$

Equilibrium Position : The system is in equilibrium if $\frac{dV}{d\theta} = 0$

$$\frac{dV}{d\theta} = W(1.5\cos\theta - 3.5\sin\theta) = 0$$

Since $W \neq 0$,

$$1.5\cos\theta - 3.5\sin\theta = 0$$
$$\theta = 23.20° = 23.2° \qquad \textbf{Ans}$$



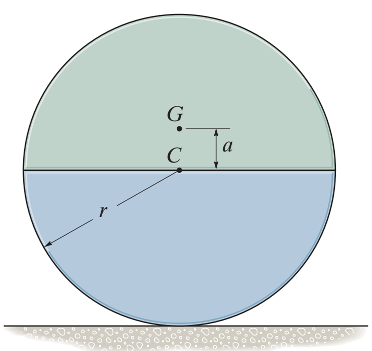
Stability :

$$\frac{d^2V}{d\theta^2} = W(-1.5\sin\theta - 3.5\cos\theta)$$

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=23.20°} = W(-1.5\sin 23.20° - 3.5\cos 23.20°) = -3.81W < 0$$

Thus, the truck is in **unstable equilibrium** at $\theta = 23.2°$ **Ans**

•11–41. The cylinder is made of two materials such that it has a mass of m and a center of gravity at point G. Show that when G lies above the centroid C of the cylinder, the equilibrium is unstable.



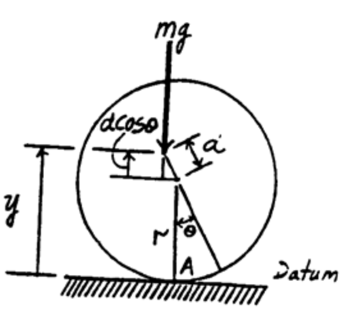
Potential Function : The datum is established at point A. Since the center of gravity of the cylinder is above the datum, its potential energy is positive. Here, $y = r + a\cos\theta$.

$$V = V_g = Wy = mg(r + a\cos\theta)$$

Equilibrium Position : The system is in equilibrium if $\frac{dV}{d\theta} = 0$.

$$\frac{dV}{d\theta} = -mga\sin\theta = 0$$

$$\sin\theta = 0 \qquad \theta = 0°.$$



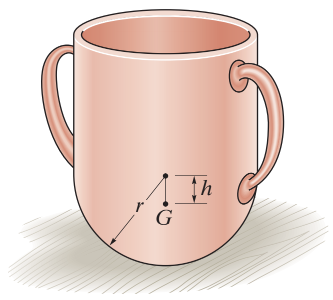
Stability :

$$\frac{d^2V}{d\theta^2} = -mga\cos\theta$$

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} = -mga\cos 0° = -mga < 0$$

Thus, the cylinder is in **unstable equilibrium** at $\theta = 0°$ (*Q.E.D.*)

11–42. The cap has a hemispherical bottom and a mass m. Determine the position h of the center of mass G so that the cup is in neutral equilibrium.



Potential Function : The datum is established at point A. Since the center of gravity of the cup is above the datum, its potential energy is positive. Here, $y = r - h\cos\theta$.

$$V = V_g = Wy = mg(r - h\cos\theta)$$

Equilibrium Position : The system is in equilibrium if $\frac{dV}{d\theta} = 0$.

$$\frac{dV}{d\theta} = mgh\sin\theta = 0$$

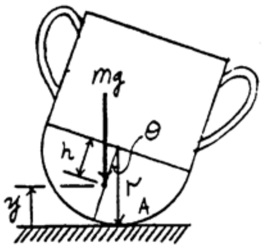
$$\sin\theta = 0 \qquad \theta = 0°.$$

Stability : To have neutral equilibrium at $\theta = 0°$, $\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} = 0.$

$$\frac{d^2V}{d\theta^2} = mgh\cos\theta$$

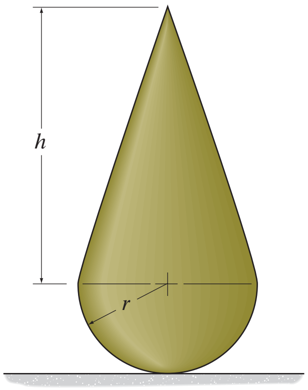
$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} = mgh\cos 0° = 0$$

$$h = 0 \qquad \textbf{Ans}$$



Note : Stable Equilibrium occurs if $h > 0\left(\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} = mgh\cos 0° > 0\right)$.

11–43. Determine the height h of the cone in terms of the radius r of the hemisphere so that the assembly is in neutral equilibrium. Both the cone and the hemisphere are made from the same material.



Potential Function: The mass of the cone and hemisphere are $m_C = \rho\left(\frac{1}{3}\pi r^2 h\right) = \frac{1}{3}\rho\pi r^2 h$ and $m_s = \rho\left(\frac{2}{3}\pi r^3\right) = \frac{2}{3}\rho\pi r^3$, where ρ is the density of the homogeneous material. With reference to the datum, Fig. a, the gravitational potential energy of the cone and hemisphere are positive since their centers of gravity are located above the datum. Here, $y_C = r + \frac{h}{4}\cos\theta$ and $y_s = r - \frac{3}{8}r\cos\theta$. Thus,

$$V = V_g = \Sigma mgy = \left(\frac{1}{3}\rho\pi r^2 h\right)(g)\left(r + \frac{h}{4}\cos\theta\right) + \frac{2}{3}\rho\pi r^3(g)\left(r - \frac{3}{8}r\cos\theta\right)$$

$$= \frac{1}{3}\rho\pi r^2 g\left(rh + \frac{h^2}{4}\cos\theta + 2r^2 - \frac{3}{4}r^2\cos\theta\right)$$

Equilibrium Configuration: Taking the first derivative of V, we have

$$\frac{dV}{d\theta} = \frac{1}{3}\rho\pi r^2 g\left(-\frac{h^2}{4}\sin\theta + \frac{3}{4}r^2\sin\theta\right) = \frac{1}{3}\rho\pi r^2 g\sin\theta\left(-\frac{h^2}{4} + \frac{3}{4}r^2\right)$$

Equilibrium requires $\frac{dV}{d\theta} = 0$. Thus,

$$\frac{1}{3}\rho\pi r^2 g\sin\theta\left(-\frac{h^2}{4} + \frac{3}{4}r^2\right) = 0$$

$$\sin\theta = 0 \quad \theta = 0°$$

Stability: The second derivative of V is

$$\frac{d^2V}{d\theta^2} = \frac{1}{3}\rho\pi r^2 g\cos\theta\left(-\frac{h^2}{4} + \frac{3}{4}r^2\right)$$

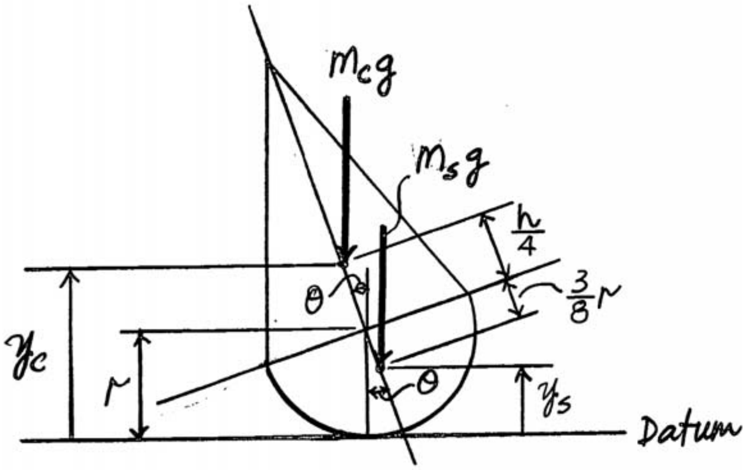
For neutral equilibrium at $\theta = 0°$, $\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} = 0$. Thus,

$$\frac{1}{3}\rho\pi r^2 g\cos 0°\left(-\frac{h^2}{4} + \frac{3}{4}r^2\right) = 0$$

Since $\frac{1}{3}\rho\pi r^2 g \neq 0$, then

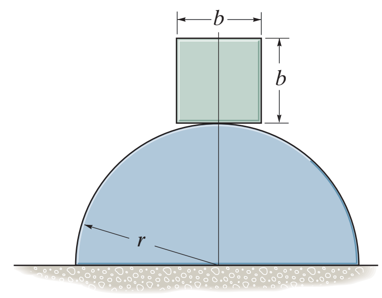
$$-\frac{h^2}{4} + \frac{3}{4}r^2 = 0$$

$$h = \sqrt{3}r \qquad \textbf{Ans.}$$



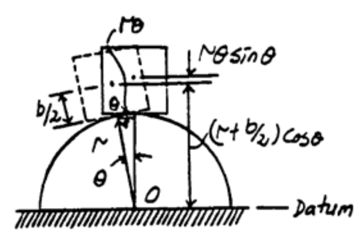
Note. The equilibrium configuration of the assembly at $\theta = 0°$ is stable if $h < \sqrt{3}r\left(\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} > 0\right)$ and is unstable if $h > \sqrt{3}r\left(\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} < 0\right)$.

***11–44.** A homogeneous block rests on top of the cylindrical surface. Derive the relationship between the radius of the cylinder, r, and the dimension of the block, b, for stable equilibrium. *Hint*: Establish the potential energy function for a small angle θ, i.e., approximate $\sin\theta \approx 0$, and $\cos\theta \approx 1 - \theta^2/2$.



Potential Function : The datum is established at point O. Since the center of gravity for the block is above the datum, its potential energy is positive. Here, $y = \left(r + \frac{b}{2}\right)\cos\theta + r\theta\sin\theta$.

$$V = W_y = W\left[\left(r + \frac{b}{2}\right)\cos\theta + r\theta\sin\theta\right] \qquad [1]$$



For small angle θ, $\sin\theta = \theta$ and $\cos\theta = 1 - \frac{\theta^2}{2}$. Then Eq. [1] becomes

$$V = W\left[\left(r + \frac{b}{2}\right)\left(1 - \frac{\theta^2}{2}\right) + r\theta^2\right]$$

$$= W\left(\frac{r\theta^2}{2} - \frac{b\theta^2}{4} + r + \frac{b}{2}\right)$$

Equilibrium Position : The system is in equilibrium if $\frac{dV}{d\theta} = 0$

$$\frac{dV}{d\theta} = W\left(r - \frac{b}{2}\right)\theta = 0 \qquad \theta = 0°$$

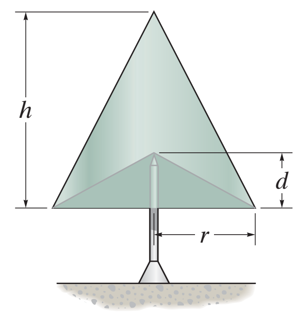
Stability : To have stable equilibrium, $\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} > 0.$

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} = W\left(r - \frac{b}{2}\right) > 0$$

$$\left(r - \frac{b}{2}\right) > 0$$

$$b < 2r \qquad \textbf{Ans}$$

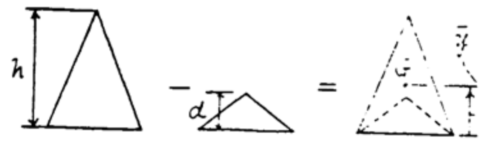
•**11–45.** The homogeneous cone has a conical cavity cut into it as shown. Determine the depth d of the cavity in terms of h so that the cone balances on the pivot and remains in neutral equilibrium.



Potential Function : The datum is established at point A. Since the center of gravity of the cone is above the datum, its potential energy is positive. Here,

$y = (\bar{y} - d)\cos\theta = \left[\frac{1}{4}(h+d) - d\right]\cos\theta = \frac{1}{4}(h-3d)\cos\theta.$

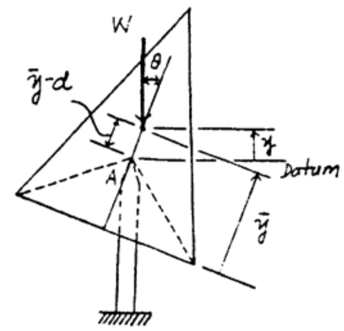
$$V = W\left[\frac{1}{4}(h-3d)\cos\theta\right] = \frac{W(h-3d)}{4}\cos\theta$$



Equilibrium Position : The system is in equilibrium if $\frac{dV}{d\theta} = 0$

$$\frac{dV}{d\theta} = -\frac{W(h-3d)}{4}\sin\theta = 0$$

$$\theta = 0 \qquad \theta = 0°$$



Stability : To have neutral equilibrium at $\theta = 0°$, $\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} = 0.$

$$\frac{d^2V}{d\theta^2} = -\frac{W(h-3d)}{4}\cos\theta$$

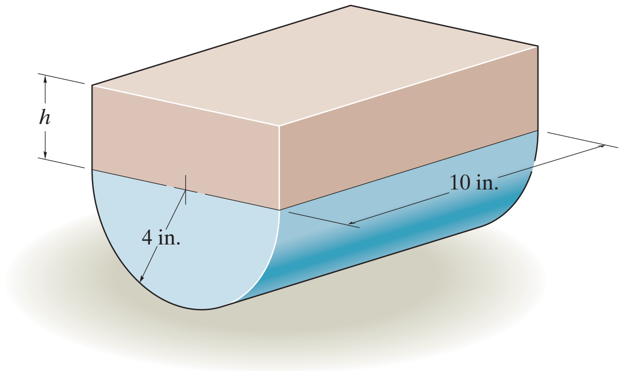
$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} = -\frac{W(h-3d)}{4}\cos 0° = 0$$

$$-\frac{W(h-3d)}{4} = 0$$

$$d = \frac{h}{3} \qquad \textbf{Ans}$$

Note : By substituting $d = \frac{h}{3}$ into Eq.[1], one realizes that the fulcrum must be at the center of gravity for neutral equilibrium.

11–46. The assembly shown consists of a semicylinder and a rectangular block. If the block weighs 8 lb and the semicylinder weighs 2 lb, investigate the stability when the assembly is resting in the equilibrium position. Set $h = 4$ in.



$$d = \frac{4(4)}{3\pi} = 1.698 \text{ in.}$$

$$V = V_g = 2(4 - 1.698\cos\theta) + 8(4 + 2\cos\theta)$$

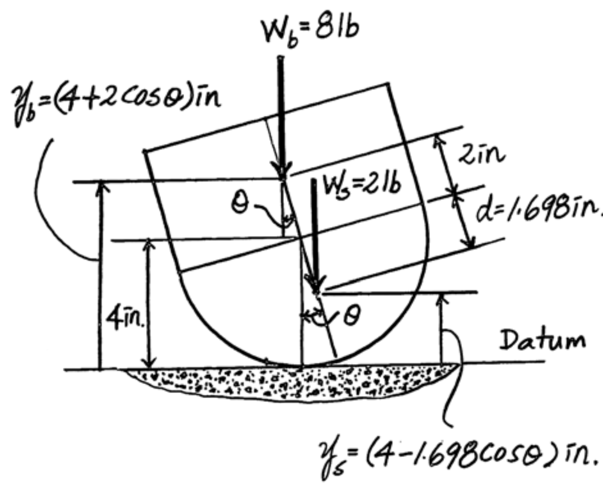
$$\frac{dV}{d\theta} = 3.395\sin\theta - 16\sin\theta = 0$$

$$\sin\theta = 0$$

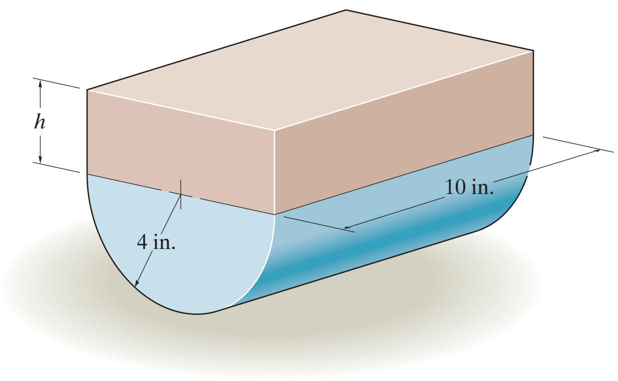
$$\theta = 0° \qquad \text{(equilibrium position)}$$

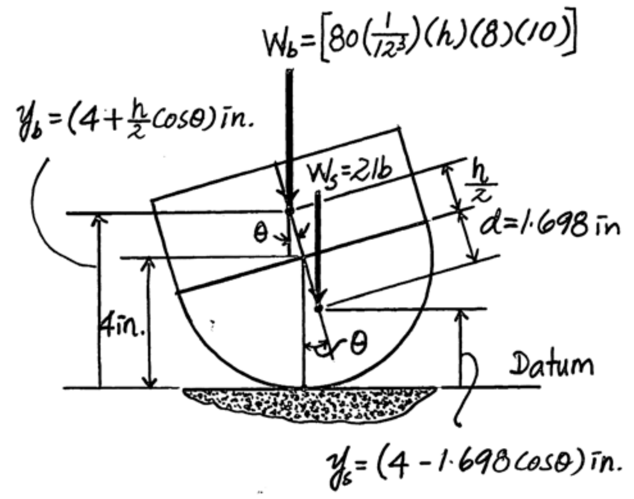
$$\frac{d^2V}{d\theta^2} = 3.395\cos\theta - 16\cos\theta$$

At $\theta = 0°$, $\quad \dfrac{d^2V}{d\theta^2} = -12.6 < 0$ Unstable Ans



11–47. The 2-lb semicylinder supports the block which has a specific weight of $\gamma = 80$ lb/ft^3. Determine the height h of the block which will produce neutral equilibrium in the position shown.





$$d = \frac{4(4)}{3\pi} = 1.698 \text{ in.}$$

$$V = V_g = 2(4 - 1.698\cos\theta) + \left[80\left(\frac{1}{12^3}\right)h(8)(10)\right]\left(4 + \frac{h}{2}\cos\theta\right)$$

$$\frac{dV}{d\theta} = 3.395\sin\theta - 1.852\,h^2\sin\theta = 0$$

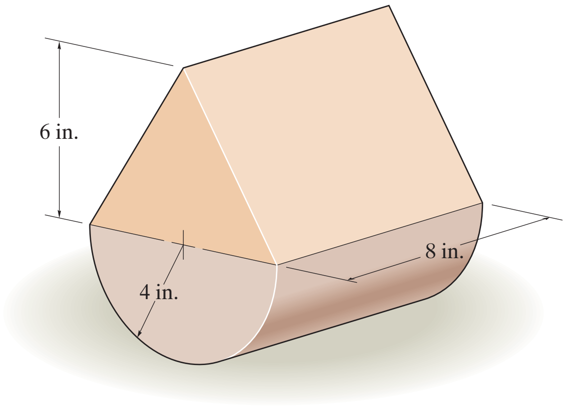
$$\sin\theta = 0$$

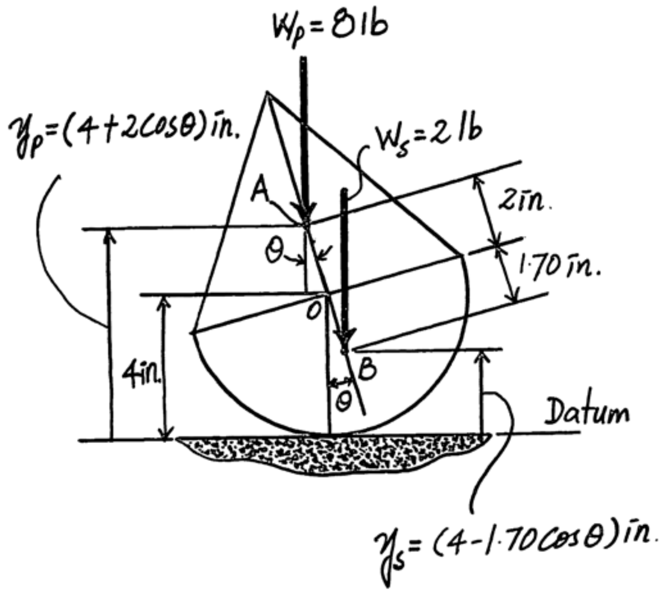
$$\theta = 0^\circ \quad \text{(equilibrium position)}$$

$$\frac{d^2V}{d\theta^2} = 3.395\cos\theta - 1.852\,h^2\cos\theta = 0$$

$$h = \sqrt{\frac{3.395}{1.852}} = 1.35 \text{ in.} \qquad \textbf{Ans}$$

***11–48.** The assembly shown consists of a semicircular cylinder and a triangular prism. If the prism weighs 8 lb and the cylinder weighs 2 lb, investigate the stability when the assembly is resting in the equilibrium position.





$$OB = \frac{4(4)}{3\pi} = 1.70 \text{ in.}$$

$$OA = \frac{1}{3}(6) = 2 \text{ in.}$$

$$V = V_g = 8(4 + 2\cos\theta) + 2(4 - 1.70\cos\theta)$$

$$V = 40 + 12.6\cos\theta$$

$$\frac{dV}{d\theta} = -12.6\sin\theta = 0$$

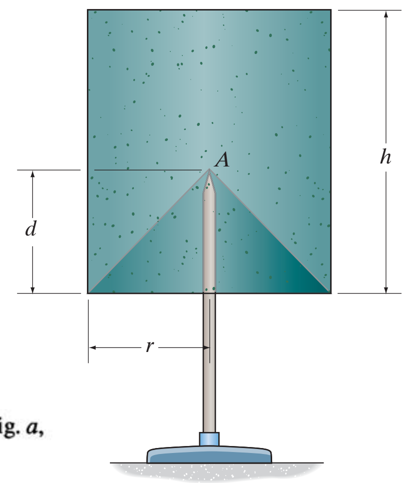
$\theta = 0°$ **Ans** (for equilibrium)

$$\frac{d^2V}{d\theta^2} = -12.6\cos\theta$$

At $\theta = 0°$,

$\frac{d^2V}{d\theta^2} = -12.6 < 0$ **unstable** **Ans**

•11–49. A conical hole is drilled into the bottom of the cylinder, and it is then supported on the fulcrum at A. Determine the minimum distance d in order for it to remain in stable equilibrium.



Potential Function: First, we must determine the center of gravity of the cylinder. By referring to Fig. a,

$$\bar{y} = \frac{\Sigma y_C m}{\Sigma m} = \frac{\frac{h}{2}(\rho\pi r^2 h) - \frac{d}{4}\left(\frac{1}{3}\rho\pi r^2 d\right)}{\rho\pi r^2 h - \frac{1}{3}\rho\pi r^2 d} = \frac{6h^2 - d^2}{4(3h - d)} \qquad (1)$$

With reference to the datum, Fig. a, the gravitational potential energy of the cylinder is positive since its center of gravity is located above the datum. Here,

$$y = (\bar{y} - d)\cos\theta = \left[\frac{6h^2 - d^2}{4(3h - d)} - d\right]\cos\theta = \left[\frac{6h^2 - 12hd + 3d^2}{4(3h - d)}\right]\cos\theta$$

Thus,

$$V = V_g = Wy = W\left[\frac{6h^2 - 12hd + 3d^2}{4(3h - d)}\right]\cos\theta$$

Equilibrium Configuration: Taking the first derivative of V,

$$\frac{dV}{d\theta} = -W\left[\frac{6h^2 - 12hd + 3d^2}{4(3h - d)}\right]\sin\theta$$

Equilibrium requires $\frac{dV}{d\theta} = 0$. Thus,

$$-W\left[\frac{6h^2 - 12hd + 3d^2}{4(3h - d)}\right]\sin\theta = 0$$

$$\sin\theta = 0 \qquad \theta = 0°$$

Stability: The second derivative of V is

$$\frac{d^2V}{d^2\theta} = -W\left[\frac{6h^2 - 12hd + 3d^2}{4(3h - d)}\right]\cos\theta$$

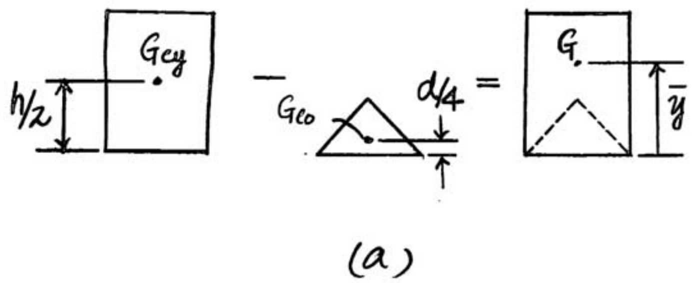
To have neutral equilibrium at $\theta = 0°$, $\left.\frac{d^2V}{d^2\theta}\right|_{\theta=0°} = 0$. Thus,

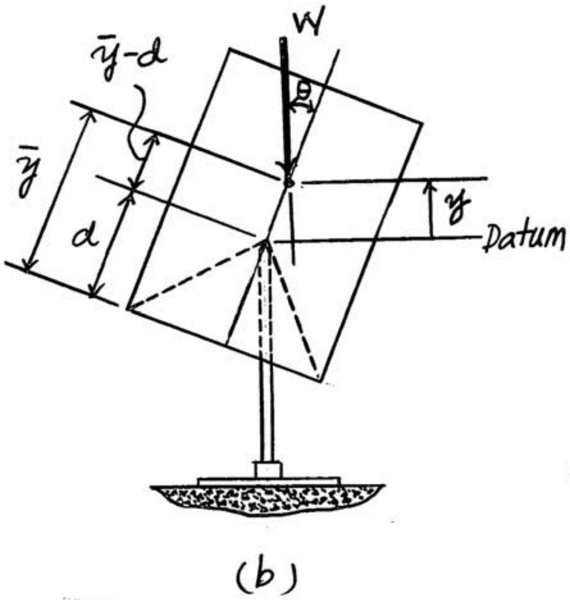
$$-W\left[\frac{6h^2 - 12hd + 3d^2}{4(3h - d)}\right]\cos 0° = 0$$

$$6h^2 - 12hd + 3d^2 = 0$$

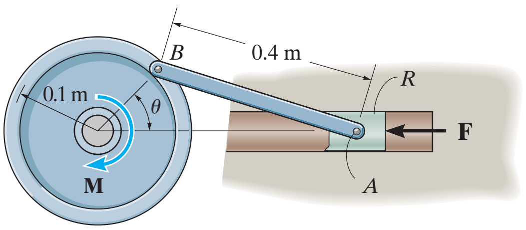
$$d = \frac{12h \pm \sqrt{(-12h)^2 - 4(3)(6h^2)}}{2(3)} = 0.5858h = 0.586h \qquad \textbf{Ans.}$$

Note: If we substitute $d = 0.5858h$ into Eq. (1), we notice that the fulcrum must be at the center of gravity for neutral equilibrium.


Gcy
h/2
Gco
d/4
G
ȳ
(a)


W
ȳ-d
θ
ȳ
d
y
Datum
(b)

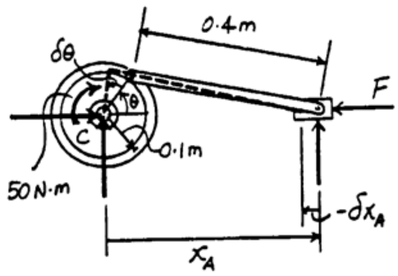
11–50. The punch press consists of the ram R, connecting rod AB, and a flywheel. If a torque of $M = 50\ \text{N}\cdot\text{m}$ is applied to the flywheel, determine the force F applied at the ram to hold the rod in the position $\theta = 60°$.



Free Body Diagram : The system has only one degree of freedom defined by the independent coordinate θ. When θ undergoes a positive displacement $\delta\theta$, only force **F** and 50 N · m couple moment do work.

Virtual Displacements : The force **F** is located from the fixed point A using the position coordinate x_A. Using the law of cosines,

$$0.4^2 = x_A^2 + 0.1^2 - 2(x_A)(0.1)\cos\theta \qquad [1]$$



Differentiating the above expression, we have

$$0 = 2x_A\,\delta x_A - 0.2\delta x_A\cos\theta + 0.2x_A\sin\theta\,\delta\theta$$

$$\delta x_A = \frac{0.2x_A\sin\theta}{0.2\cos\theta - 2x_A}\delta\theta \qquad [2]$$

Virtual - Work Equation : When point A undergoes positive virtual displacement δx_A, force **F** does negative work. The 50 N · m couple moment does negative work when the flywheel undergoes a positive virtual rotation $\delta\theta$.

$$\delta U = 0; \qquad -F\delta x_A - 50\delta\theta = 0 \qquad [3]$$

Substituting Eq. [2] into [3] yields

$$\left(-\frac{0.2x_A\sin\theta}{0.2\cos\theta - 2x_A}F - 50\right)\delta\theta = 0$$

Since $\delta\theta \neq 0$, then

$$-\frac{0.2x_A\sin\theta}{0.2\cos\theta - 2x_A}F - 50 = 0$$

$$F = -\frac{50(0.2\cos\theta - 2x_A)}{0.2x_A\sin\theta} \qquad [4]$$

At the equilibrium position, $\theta = 60°$. Substituting into Eq. [1], we have

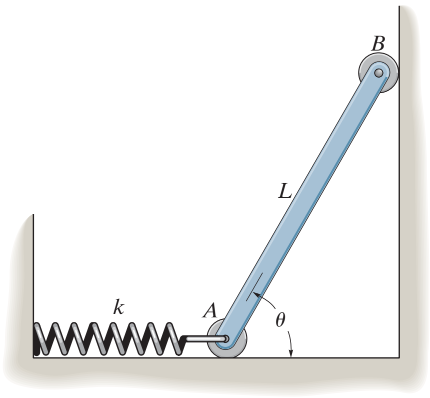
$$0.4^2 = x_A^2 + 0.1^2 - 2(x_A)(0.1)\cos 60°$$

$$x_A = 0.4405\ \text{m}$$

Substituting the above results into Eq. [4], we have

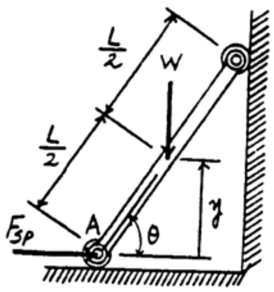
$$F = -\frac{50[0.2\cos 60° - 2(0.4405)]}{0.2(0.4405)\sin 60°} = 512\ \text{N} \qquad \textbf{Ans}$$

11–51. The uniform rod has a weight W. Determine the angle θ for equilibrium. The spring is uncompressed when $\theta = 90°$. Neglect the weight of the rollers.



Potential Function : The datum is established at point A. Since the center of gravity of the beam is above the datum, its potential energy is positive. Here, $y = \frac{L}{2}\sin\theta$ and the spring compresses $x = L\cos\theta$.

$$V = V_e + V_g$$
$$= \frac{1}{2}kx^2 + Wy$$
$$= \frac{1}{2}(k)(L\cos\theta)^2 + W\left(\frac{L}{2}\sin\theta\right)$$
$$= \frac{kL^2}{2}\cos^2\theta + \frac{WL}{2}\sin\theta$$



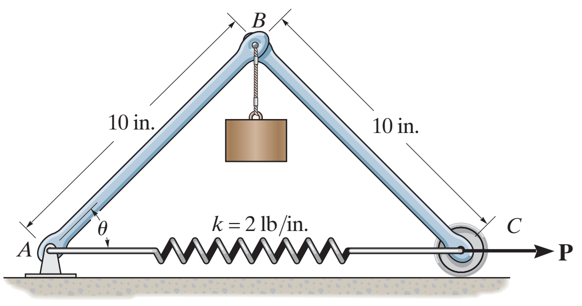
Equilibrium Position : The system is in equilibrium if $\frac{dV}{d\theta} = 0$.

$$\frac{dV}{d\theta} = -kL^2\sin\theta\cos\theta + \frac{WL}{2}\cos\theta = 0$$
$$\cos\theta\left(-kL^2\sin\theta + \frac{WL}{2}\right) = 0$$

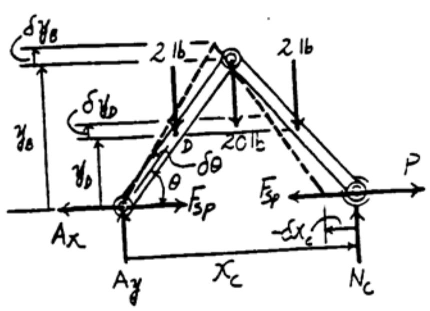
Solving,

$$\theta = 90° \quad \text{or} \quad \sin^{-1}\left(\frac{W}{2kL}\right) \qquad \textbf{Ans}$$

***11–52.** The uniform links AB and BC each weigh 2 lb and the cylinder weighs 20 lb. Determine the horizontal force P required to hold the mechanism at $\theta = 45°$. The spring has an unstretched length of 6 in.



Free Body Diagram : The system has only one degree of freedom defined by the independent coordinate θ. When θ undergoes a positive displacement $\delta\theta$, only the spring force F_{sp}, the weight of links (2 lb), 20 lb force and force **P** do work.



Virtual Displacements : The positions of points B, D and C are measured from the fixed point A using position coordinates y_B, y_D and x_C respectively.

$$y_B = 10\sin\theta \qquad \delta y_B = 10\cos\theta\,\delta\theta \qquad [1]$$

$$y_D = 5\sin\theta \qquad \delta y_D = 5\cos\theta\,\delta\theta \qquad [2]$$

$$x_C = 2(10\cos\theta) \qquad \delta x_C = -20\sin\theta\,\delta\theta \qquad [3]$$

Virtual - Work Equation : When points B, D and C undergo positive virtual displacements δy_B, δy_D and δx_C, spring force F_{sp} that acts at point C, the weight of links (2 lb) and 20 lb force do negative work while force **P** does positive work.

$$\delta U = 0; \qquad -F_{sp}\,\delta x_C - 2(2\delta y_D) - 20\delta y_B + P\delta x_C = 0 \qquad [4]$$

Substituting Eqs.[1], [2] and [3] into [4] yields

$$(20F_{sp}\sin\theta - 20P\sin\theta - 220\cos\theta)\,\delta\theta = 0 \qquad [5]$$

However, from the spring formula, $F_{sp} = kx = 2[2(10\cos\theta) - 6]$ $= 40\cos\theta - 12$. Substituting this value into Eq.[5] yields

$$(800\sin\theta\cos\theta - 240\sin\theta - 220\cos\theta - 20P\sin\theta)\,\delta\theta = 0$$

Since $\delta\theta \neq 0$, then

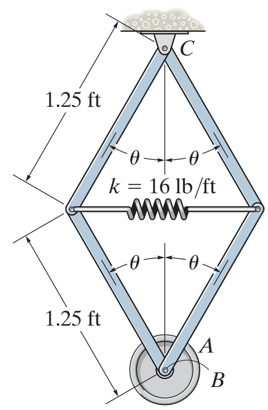
$$800\sin\theta\cos\theta - 240\sin\theta - 220\cos\theta - 20P\sin\theta = 0$$

$$P = 40\cos\theta - 11\cot\theta - 12$$

At the equilibrium position, $\theta = 45°$. Then

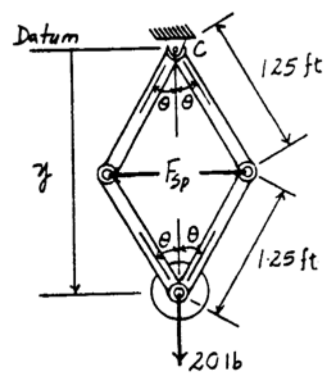
$$P = 40\cos 45° - 11\cot 45° - 12 = 5.28 \text{ lb} \qquad \textbf{Ans}$$

•11–53. The spring attached to the mechanism has an unstretched length when $\theta = 90°$. Determine the position θ for equilibrium and investigate the stability of the mechanism at this position. Disk A is pin connected to the frame at B and has a weight of 20 lb.



Potential Function : The datum is established at point C. Since the center of gravity of the disk is below the datum, its potential energy is negative. Here, $y = 2(1.25\cos\theta) = 2.5\cos\theta$ ft and the spring compresses $x = (2.5 - 2.5\sin\theta)$ ft.

$$\begin{aligned} V &= V_e + V_g \\ &= \frac{1}{2}kx^2 - Wy \\ &= \frac{1}{2}(16)(2.5 - 2.5\sin\theta)^2 - 20(2.5\cos\theta) \\ &= 50\sin^2\theta - 100\sin\theta - 50\cos\theta + 50 \end{aligned}$$



Equilibrium Position : The system is in equilibrium if $\dfrac{dV}{d\theta} = 0$.

$$\frac{dV}{d\theta} = 100\sin\theta\cos\theta - 100\cos\theta + 50\sin\theta = 0$$
$$\frac{dV}{d\theta} = 50\sin 2\theta - 100\cos\theta + 50\sin\theta = 0$$

Solving by trial and error,

$$\theta = 37.77° = 37.8° \qquad \textbf{Ans}$$

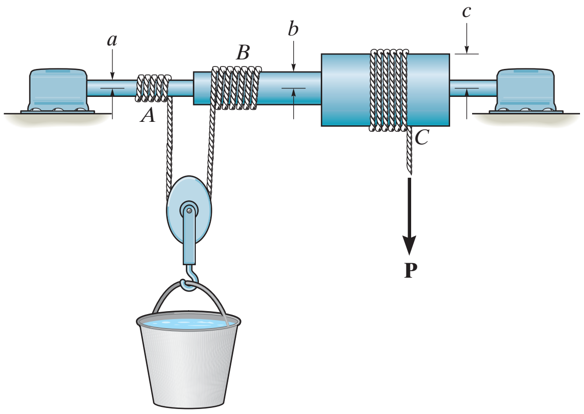
Stability :

$$\frac{d^2V}{d\theta^2} = 100\cos 2\theta + 100\sin\theta + 50\cos\theta$$

$$\begin{aligned} \left.\frac{d^2V}{d\theta^2}\right|_{\theta = 37.77°} &= 100\cos 75.54° + 100\sin 37.77° + 50\cos 37.77° \\ &= 125.7 > 0 \end{aligned}$$

Thus, the system is in **stable equilibrium** at $\theta = 37.8°$ **Ans**

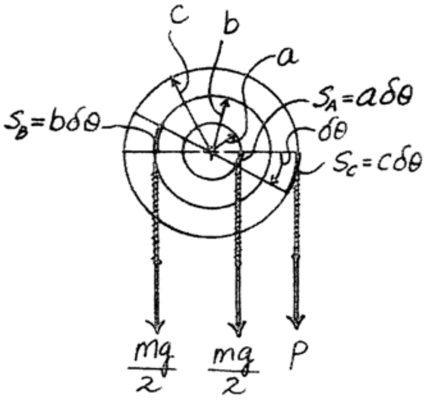
11–54. Determine the force P that must be applied to the cord wrapped around the drum at C which is necessary to lift the bucket having a mass m. Note that as the bucket is lifted, the pulley rolls on a cord that winds up on shaft B and unwinds from shaft A.



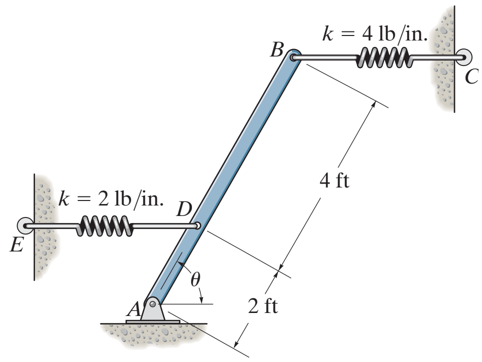
As shaft rotates $\delta\theta$

$$\delta U = 0; \quad P(c)\,\delta\theta - \frac{mg}{2}(b\,\delta\theta) + \frac{mg}{2}(a\,\delta\theta) = 0$$

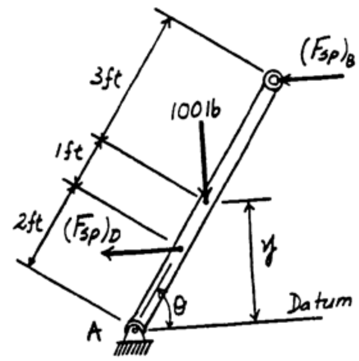
$$P = \left(\frac{b-a}{2c}\right)mg \quad \text{Ans}$$



11–55. The uniform bar AB weighs 100 lb. If both springs DE and BC are unstretched when $\theta = 90°$, determine the angle θ for equilibrium using the principle of potential energy. Investigate the stability at the equilibrium position. Both springs always remain in the horizontal position due to the roller guides at C and E.



Potential Function : The datum is established at point A. Since the center of gravity of the beam is above the datum, its potential energy is positive. Here, $y = (3\sin\theta)$ ft, the spring at D stretches $x_D = (2\cos\theta)$ ft and the spring at B compresses $x = (6\cos\theta)$ ft.



$$\begin{aligned} V &= V_e + V_g \\ &= \Sigma\frac{1}{2}kx^2 + Wy \\ &= \frac{1}{2}(24)(2\cos\theta)^2 + \frac{1}{2}(48)(6\cos\theta)^2 + 100(3\sin\theta) \\ &= 912\cos^2\theta + 300\sin\theta \end{aligned}$$

Equilibrium Position : The system is in equilibrium if $\frac{dV}{d\theta} = 0$.

$$\frac{dV}{d\theta} = -1824\sin\theta\cos\theta + 300\cos\theta = 0$$

$$\frac{dV}{d\theta} = -912\sin 2\theta + 300\cos\theta = 0$$

Solving,

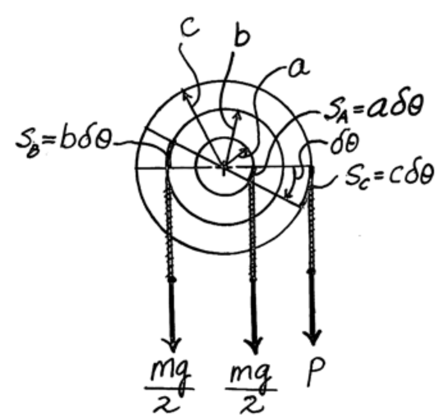
$$\theta = 90° \quad \text{or} \quad \theta = 9.467° = 9.47° \qquad \textbf{Ans}$$

Stability :

$$\frac{d^2V}{d\theta^2} = -1824\cos 2\theta - 300\sin\theta$$

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=90°} = -1824\cos 180° - 300\sin 90° = 1524 > 0$$

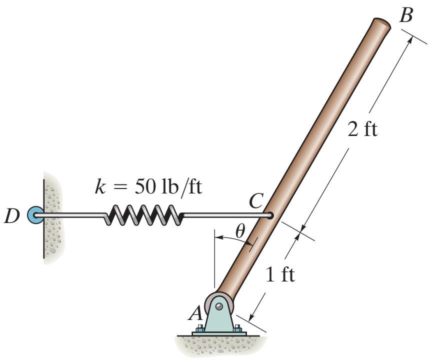
Thus, the system is in **stable equilibrium** at $\theta = 90°$ **Ans**



$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=9.467°} = -1824\cos 18.933° - 300\sin 9.467° = -1774.7 < 0$$

Thus, the system is in **unstable equilibrium** at $\theta = 9.47°$ **Ans**

***11–56.** The uniform rod AB has a weight of 10 lb. If the spring DC is unstretched when $\theta = 0°$, determine the angle θ for equilibrium using the principle of virtual work. The spring always remains in the horizontal position due to the roller guide at D.



$y_w = 1.5\cos\theta \qquad \delta y_w = -1.5\sin\theta\,\delta\theta$

$x_F = 1\sin\theta \qquad \delta x_F = \cos\theta\,\delta\theta$

$\delta U = 0; \qquad -W\delta y_w - F_s\,\delta x_F = 0$

$-10(-1.5\sin\theta\delta\theta) - F_s(\cos\theta\,\delta\theta) = 0$

$\delta\theta(15\sin\theta - F_s\cos\theta) = 0$

Since $\delta\theta \neq 0$

$15\sin\theta - F_s\cos\theta = 0 \qquad (1)$

$F_s = kx \qquad$ where $\quad x = 1\sin\theta \qquad (2)$

$F_s = 50(\sin\theta) = 50\sin\theta$

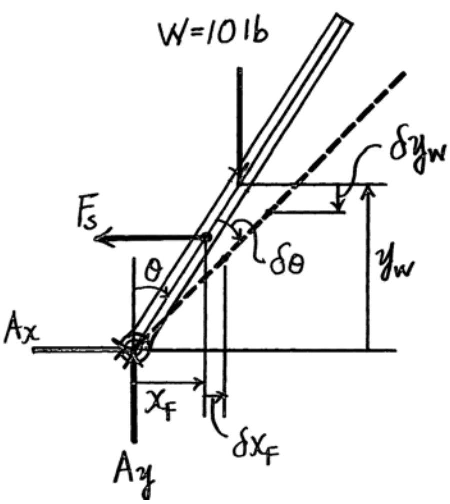
Substituting Eq. (2) into (1) yields :

$15\sin\theta - (50\sin\theta)\cos\theta = 0$

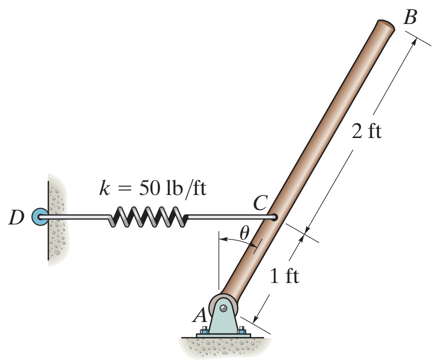
$\sin\theta(15 - 50\cos\theta) = 0$

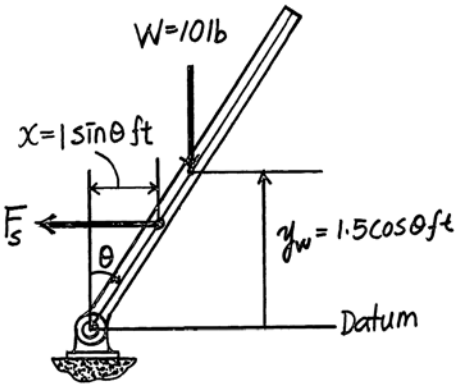
$\sin\theta = 0 \qquad \theta = 0° \qquad$ **Ans**

$15 - 50\cos\theta = 0 \qquad \theta = 72.5° \qquad$ **Ans**



•11–57. Solve Prob. 11–56 using the principle of potential energy. Investigate the stability of the rod when it is in the equilibrium position.





$$V = V_e + V_g = \frac{1}{2}(50)(\sin\theta)^2 + 10(1.5\cos\theta)$$

$$= 25\sin^2\theta + 15\cos\theta$$

$$\frac{dV}{d\theta} = 0$$

$$\frac{dV}{d\theta} = 50\sin\theta\cos\theta - 15\sin\theta = 0$$

$$\sin\theta(50\cos\theta - 15) = 0$$

$\sin\theta = 0$ $\qquad \theta = 0°$ $\qquad$ **Ans**

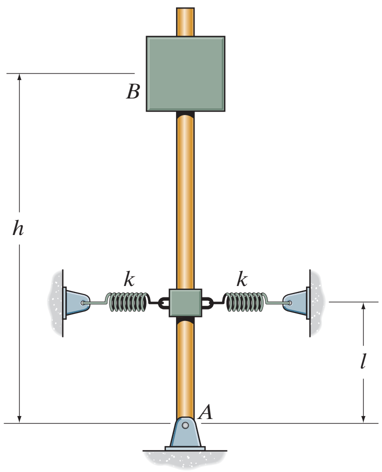
$50\cos\theta - 15 = 0$ $\qquad \theta = 72.5°$ $\qquad$ **Ans**

$$\frac{d^2V}{d\theta^2} = 50\cos 2\theta - 15\cos\theta$$

At $\theta = 0°$, $\quad \frac{d^2V}{d\theta^2} = 50\cos 0° - 15\cos 0° = 35 > 0$ $\quad$ **stable** $\quad$ **Ans**

At $\theta = 72.5°$, $\quad \frac{d^2V}{d\theta^2} = 50\cos 145° - 15\cos 72.5° = -45.5 < 0$ $\quad$ **unstable** $\quad$ **Ans**

11–58. Determine the height h of block B so that the rod is in neutral equilibrium. The springs are unstretched when the rod is in the vertical position. The block has a weight W.



Potential Function: With reference to the datum, Fig. *a*, the gravitational potential energy of block B is positive since its center of gravity is located above the datum. Here, the rod is tilted a small angle θ. Thus, $y = h\cos\theta$. For a small angle θ, $\cos\theta \approx 1 - \frac{\theta^2}{2}$.

Thus,

$$V_g = Wy = Wh\left(1 - \frac{\theta^2}{2}\right)$$

The elastic potential energy of each spring can be computed using $V_e = \frac{1}{2}ks^2$. Since θ is small, $s \approx l\theta$. Thus,

$$V_e = 2\left[\frac{1}{2}k(l\theta)^2\right] = kl^2\theta^2$$

The total potential energy of the system is

$$V = V_g + V_e = Wh\left(1 - \frac{\theta^2}{2}\right) + kl^2\theta^2$$

Equilibrium Configuration: Taking the first derivative of V,

$$\frac{dV}{d\theta} = -Wh\theta + 2kl^2\theta = \theta(-Wh + 2kl^2)$$

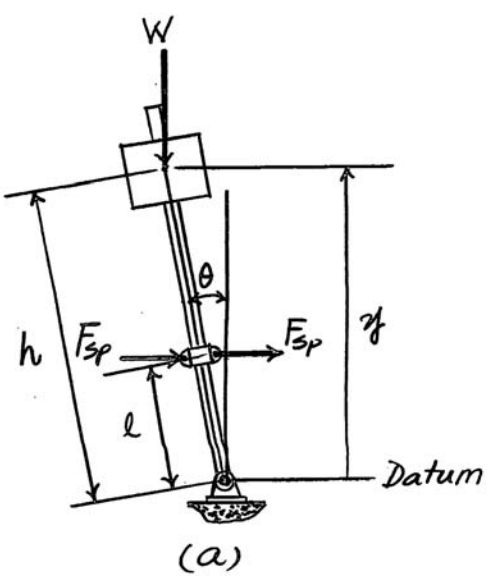
Equilibrium requires $\frac{dV}{d\theta} = 0$. Thus,

$$\theta(-Wh + 2kl^2) = 0$$

$$\theta = 0°$$

Stability: The second derivative of V is

$$\frac{d^2V}{d\theta^2} = -Wh + 2kl^2$$



To have neutral equilibrium at $\theta = 0°$, $\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} = 0$. Thus,

$$-Wh + 2kl^2 = 0$$

$$h = \frac{2kl^2}{W} \qquad \textbf{Ans.}$$

Note: The equilibrium configuration of the system at $\theta = 0°$ is stable if $h < \frac{2kl^2}{W}\left(\frac{d^2V}{d\theta^2} > 0\right)$ and is unstable if $h > \frac{2kl^2}{W}\left(\frac{d^2V}{d\theta^2} < 0\right)$.